Lectures on Game Theory and Social Choice

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Games as *data structures* to model *multiagent systems* i.e. *interactions among intentional systems*

A game typically consists of a *game form* and a *profile of player-types:*

- the game form describes the rules of the game
- the player-types include information about the *behavioural attitudes of players towards outcomes* (BAPO) (those attitudes are most typically coded in a profile of binary preference relations)

REMARK: Preference relations are best regarded as *revealed preferences* i.e. a coding of agents' choice behaviour that is indeed testable with choice-databases.

Three Basic Game Formats:

- Strategic Format: [Strategic Game Form =Who Can Do What and How]+BAPO
- Extensive Format: [Extensive Game Form=Who Can Do What, How and When]+BAPO
- Coalitional Format: [Coalitional Game Form=Who Can Do What]+BAPO

A **Game in Strategic Form** is a structure
$G = (N, X, (S_i)_{i\in N}, h, (t_i(\succcurlyeq_i))_{i\in N})$ where

N set of *players*, $N \neq \varnothing$ and usually finite ($N = \{1, ..., n\}$)

X set/space of *outcomes*

S_i nonempty set *strategies* of player i, $i \in N$

$h : \prod_{i\in N} S_i \to X$ *strategic outcome function*

$t_i(\succcurlyeq_i)$ *type* of player i, $i \in N$, including his/her *preference relation* $\succcurlyeq_i \subseteq X \times X$

(usually taken to be a *total preorder* i.e.

reflexive: $x \succcurlyeq_i x$ for each $x \in X$,

transitive: *if* $x \succcurlyeq_i y$ and $y \succcurlyeq_i z$ then $x \succcurlyeq_i z$ for any $x, y, z \in X$

connected: $x \succcurlyeq_i y$ or $y \succcurlyeq_i x$ for any $x, y \in X$, $x \neq y$,

and in that case representable by

an *utility function* $u_i : X \to \mathbb{R}$ such that for any $x, y \in X$, $u_i(x) \geq u_i(y)$ if and only if $x \succcurlyeq_i y$)

A standard 'contrapted' version of the strategic format
$\widehat{G} = (N, (S_i)_{i\in N}, (\widehat{\succcurlyeq}_i)_{i\in N})$ where

N set of players

S_i nonempty set of strategies of player i, $i \in N$

$\widehat{\succcurlyeq}_i \subseteq \prod_{i\in N} S_i \times \prod_{i\in N} S_i$ the preference relation of player i, $i \in N$, typically a *total preorder*, representable by an utility function $f_i : \prod_{i\in N} S_i \to \mathbb{R}$.

REMARK

$\widehat{G}$ is indeed isomorphic to a *special case* of
$G = (N, X, (S_i)_{i\in N}, h, (t_i(\succcurlyeq_i))_{i\in N})$
where

$X := \prod_{i\in N} S_i$,

$h := id : \prod_{i\in N} S_i \to \prod_{i\in N} S_i$

i.e.

$\widehat{G} \simeq G' = (N, \prod_{i\in N} S_i, (S_i)_{i\in N}, id, (\widehat{\succcurlyeq}_i)_{i\in N})$.

EXAMPLES

2×2 BIMATRIX GAMES

$G = (N = \{1,2\}, X = \{a,b,c,d\}, (S_i = \{s_i, t_i\})_{i \in \{1,2\}}, h, (\succcurlyeq_i)_{i \in \{1,2\}})$

with $h((s_1, s_2)) = a$, $h((s_1, t_2)) = b$,

$h((t_1, s_2)) = c$, $h((t_1, t_2)) = d$,

representable by the matrix- game form

1/2	t_1	t_2
s_1	a	b
s_2	c	d

with preferences $\succcurlyeq_1 \subseteq \{a,b,c,d\}^2$, $\succcurlyeq_2 \subseteq \{a,b,c,d\}^2$, total preorders on $\{a,b,c,d\}$.

The *degree of conflict* between the preferences of players can be measured through the *Kemeny distance* (namely counting the number of simple permutations between adjacent outcomes to make one of the preferences identical to the other).

Battle of the Sexes (BS)

general description:[**Bargaining game with minimal conflict of interest and no focal point]**

1/2	b_2	s_2
b_1	a	b
s_1	c	d

1/2	b_2	s_2
b_1	3; 2	1; 1
s_1	0; 0	2; 3

$\succcurlyeq_1 := (adbc) \longrightarrow u_1$

$\succcurlyeq_2 := (dabc) \longrightarrow u_2$

degree of conflict: 1 out of 6 ($ad \rightarrow da$)

Legenda (here and henceforth): outcomes in decreasing preference order

Legenda for the canonical interpretation: b_i=attending the Bach concert, s_i=attending the Strawinsky concert

Prisoner Dilemma (PD)
general description [**Nonstrictly-competitive coordination-proof game]**

1/2	c_2	d_2
c_1	a	b
d_1	c	d

1/2	c_2	d_2
c_1	2; 2	0; 3
d_1	3; 0	1; 1

$\succcurlyeq_1 := (cadb) \longrightarrow u_1 \uparrow$ (payoffs of player 1)

$\succcurlyeq_2 := (badc) \longrightarrow u_2 \uparrow$ (payoffs of player 2)

degree of conflict: 5 out of 6 (a single agreement on pair (a, d))

Legenda for the canonical interpretation: c_i =cooperate (with the other player), d_i =defect

Stag Hunt (SH) [Local/regional variation : Wild Boar Hunt]
general description:**[Common interest coordination game with strong conflict of interest]**

1/2	s_2	h_2
s_1	a	b
h_1	c	d

1/2	s_2	h_2
s_1	3; 3	0; 2
h_1	2; 0	1; 1

$\succcurlyeq_1 := (acdb) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (abdc) \longrightarrow u_2 \uparrow$

degree of conflict: 3 out of 6

Legenda for the canonical interpretation: s_i =stag hunting, h_i =hare hunting

Hawk-Dove (HD)

general description **[Bargaining game with common threat-outcome]**

1/2	d_2	h_2
d_1	a	b
h_1	c	d

1/2	d_2	h_2
d_1	2; 2	1; 3
h_1	3; 1	0; 0

$\succcurlyeq_1 := (cabd) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (bacd) \longrightarrow u_2 \uparrow$

degree of conflict: 3 out of 6

Legenda for the canonical interpretation: d_i =dovish behaviour (e.g. stop at the crossroad), h_i =hawkish behaviour (e.g. cross the road)

Asymmetric Prisoner Dilemma (APD)

general description **[Coordination-proof leadership game]**

1/2	c_2	d_2
c_1	a	b
d_1	c	d

1/2	c_2	d_2
c_1	2; 3	0; 2
d_1	3; 0	1; 1

$\uparrow$

$\succcurlyeq_1 := (cadb) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (abdc) \longrightarrow u_2$

degree of conflict: 4 out of 6

Legenda for the canonical interpretation: same as for Prisoner Dilemma

Winner-Loser (WL)

general description: **[Strictly competitive game with asymmetric player roles]**

1/2	s_2	t_2
s_1	a	b
t_1	c	d

1/2	s_2	t_2
s_1	2; 1	3; 0
t_1	1; 2	0; 3

$\succcurlyeq_1 := (bacd) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (dcab) \longrightarrow u_2 \uparrow$

degree of conflict: 6 out of 6 (maximum)

Matching Pennies (MP)

general description: **[Chaotic strictly competitive game]**

1/2	h_2	t_2
h_1	a	b
t_1	c	d

1/2	h_2	t_2
h_1	1; 2	3; 0
t_1	2; 1	0; 3

$\succcurlyeq_1 := (bcad) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (dacb) \longrightarrow u_2 \uparrow$

degree of conflict: 6 out of 6 (maximum)

Legenda for the canonical interpretation: h_i =choose head, t_i =choose tail

Round Dance (RD)

general description: **[Chaotic non-strictly-competitive game]**

1/2	s_2	t_2
s_1	a	b
t_1	c	d

1/2	s_2	t_2
s_1	1; 0	2; 1
t_1	0; 3	3; 2

$\succcurlyeq_1 := (dbac) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (cdba) \longrightarrow u_2 \uparrow$

degree of conflict: 3 out of 6

Invisible Hand (IH)

general description: **[Self-organized game with spontaneous order]**

1/2	c_2	d_2
c_1	a	b
d_1	c	d

1/2	c_2	d_2
c_1	2; 2	3; 0
d_1	0; 3	1; 1

$\uparrow$

$\succcurlyeq_1 := (badc) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (cadb) \longrightarrow u_2$

degree of conflict: 5 out of 6

Legenda for the canonical interpretation: same as for PD

Soft Assurance (SA)

general description:[**Common interest coordination game with weak conflict of interest]**

1/2	c_2	d_2
c_1	a	b
d_1	c	d

1/2	c_2	d_2
c_1	3; 3	0; 1
d_1	1; 0	2; 2

$\succcurlyeq_1 := (adcb) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (adbc) \longrightarrow u_2 \uparrow$

degree of conflict: 1 out of 6

Legenda for the canonical interpretation: c_i =cooperate, d_i =defect

Team Coordination (TC)

general description:[**Common interest coordination game with no conflict of interest: an Ordinal Potential game]**

1/2	s_2	t_2
s_1	a	b
t_1	c	d

1/2	s_2	t_2
s_1	3; 3	0; 0
t_1	1; 1	2; 2

$$\succcurlyeq_1 := (adcb) \longrightarrow u_1 \uparrow$$

$$\succcurlyeq_2 := (adcb) \longrightarrow u_2 \uparrow$$

degree of conflict: 0 out of 6

Legenda for the canonical interpretation: s_i =apply first protocol, t_i =apply second protocol

Hierarchical Coordination (HC)

general description:[**Coordination game with an effective leadership]**

1/2	s_2	t_2
s_1	a	b
t_1	c	d

1/2	s_2	t_2
s_1	2; 3	3; 1
t_1	1; 0	0; 2

$$\uparrow$$

$\succcurlyeq_1 := (bacd) \longrightarrow u_1 \uparrow$

$\succcurlyeq_2 := (adbc) \longrightarrow u_2$

degree of conflict: 3 out of 6

EXAMPLE:

Rock Scissors Paper

1/2	R	S	P
R	$0;0$	$1;-1$	$-1;1$
S	$-1;1$	$0;0$	$1;-1$
P	$1;-1$	$-1;1$	$0;0$

...... or **Uta Stansburiana Lizard Game**

1/2	O	B	S
O	$0;0$	$1;-1$	$-1;1$
B	$-1;1$	$0;0$	$1;-1$
S	$1;-1$	$-1;1$	$0;0$

EXAMPLE

'**Competition by differentiation**' bimatrix game :

1/2	L	M	H
L	0; 0	0; 2	1; 3
M	2; 0	1; 1	2; 0
H	3; 1	0; 2	0; 0

Legenda for the canonical interpretation: L =low quality production, M =medium quality production, H =high quality production

EXAMPLE

Majority voting game form with pseudorandom endogenous president selection

Game form $\Gamma^{maj} = (N, X, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{maj})$,
where

$N = \{1, ..., n\}$,

$X = \{x_1, ...x_m\}$

$h^{maj} : (X \times \mathbb{Z}_+)^N \to X$, is defined by the following rules:

for each $s^N = ((x_1, z_1), ..., (x_n, z_n)) \in (L_X \times \mathbb{Z}_+)^N$,

$h^{maj}(s^N) = x$ if there exists precisely one $x \in X$ such that $|\{i \in N : x_i = x\}| \geq \frac{n}{2}$,

and $x_{i^*(s^N)}$ otherwise,

where $i^*(s^N) = \sum_{i \in N} z_i (\mathrm{mod}\, n))$.

For any preference profile $(\succcurlyeq_i)_{i \in N}$ of total preorders with an unique maximum,

$G^{maj} = (\Gamma^{maj}, (\succcurlyeq_i)_{i \in N}) = (N, X, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{maj}, (\succcurlyeq_i)_{i \in N})$

is a *majority voting game in strategic form*.

EXAMPLE

Random Dictatorship voting game form

Game form $\Gamma^{rd} = (N, X, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{rd})$,
where

$N = \{1, ..., n\}$,

$X = \{x_1, ...x_m\}$

$h^{rd}(s^N) = x_{i^*(s^N)}$, with $i^*(s^N) = \sum_{i \in N} z_i (\mathrm{mod}\, n))$.

For any preference profile $(\succcurlyeq_i)_{i \in N}$ of total preorders with an unique maximum,

$G^{rd} = (\Gamma^{rd}, (\succcurlyeq_i)_{i \in N}) = (N, X, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{rd}, (\succcurlyeq_i)_{i \in N})$

is a *random dictatorship voting game in strategic form*.

EXAMPLE

Guess the Average prize-contest game

$$G^{GA} = (N, X = \{0,1\}^N, (S_i = [1, ..., 999])_{i \in N}, h^{W(2/3)}, (\succcurlyeq_i)_{i \in N})$$

each player selects a number in the range $[1, ..., 999]$.
For any $i \in N = \{1, ..., n\}$, and all $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X$,

$x \succcurlyeq_i y$ if and only if $x_i \geq y_i$

The winners at strategy profile $s = (s_1, ..., s_n)$ are the players who choose the number(s) closest to $\frac{2}{3}(\frac{1}{n} \sum_{i \in N} s_i)$.

If a player $i \in N$ is a winner at s i.e.

$$|s_i - \frac{2}{3} \frac{\Sigma_{j \in N} s_j}{n}| \leq |s_j - \frac{2}{3} \frac{\Sigma_{j \in N} s_j}{n}| \text{ for all } j \in N,$$

then $(h^{W(2/3)}(s))_i = 1$.

EXAMPLE

Sealed Bid Second-Price Auctions for Single Items with Private Values (Vickrey Auctions)

Strategic game form:
$\Gamma^{SP} = \left(N = \{1, ..., n\}, \mathbb{R}_+^{2n}, (S_i = \mathbb{R}_+)_{i \in N}, h^r\right)$
where the strategic outcome function

$h^r := a(.) \times p(.) : \mathbb{R}_+^n \to \mathbb{R}_+^{2n}$ can be regarded as the combination of an

allocation rule $a(.) : \mathbb{R}_+^n \to \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \leq 1 \right\}$

and a

payment rule

$p(.) : \mathbb{R}_+^n \to \{p \in \mathbb{R}_+^n : \text{if } p_i > 0 \text{ then } p_i \geq r \text{ and } p_j = 0 \text{ for all } j \neq i\}$
($r \in \mathbb{R}_+$ is the *reserve price* of the seller).

The allocation rule is defined as follows: for each profile $s = (s_1, ..., s_n)$ of *bids*

$a_{i^*}(s) = 1$ if and only if
$i^* = \inf \left\{ i \in N : s_i \geq r \text{ and } s_i = \sup_{1 \leq j \leq n} s_j \right\}$.

(continues)

(continues)

The payment rule is defined as follows : for each profile $s = (s_1, ..., s_n)$ of *bids*

$$p_i(s) = s_{j^*} \text{ where}$$
$$j^* = \inf \left\{ j \in N : j \neq i,\ s_j \geq r \text{ and } s_j = \sup_{1 \leq j \leq n, j \neq i} s_j \right\}$$
$$\text{if } a_i(s) = 1 \text{ and such a } j^* \text{ exists}$$
$$= s_i \text{ if } a_i(s) = 1 \text{ and such a } j^* \text{ does not exist}$$
$$= 0 \text{ otherwise.}$$

Notice that both $a[\mathbb{R}^n_+] \subset \mathbb{R}^n_+$ and $p[\mathbb{R}^n_+] \subset \mathbb{R}^n_+$ hence h^r is *not* onto.

The players' preferences $\succcurlyeq_i$ are *quasilinear* and representable by quasilinear utility functions u_i with

$$u_i(h^r(s)) = v_i - p_i(s) \text{ (where } v_i \text{ is the \textit{true value} of the item to } i\text{)}$$
$$\text{if } a_i(s) = 1, \text{ and}$$
$$= 0 \text{ otherwise.}$$

Solution Rules for Games in Strategic Form

A *solution rule* (for games in strategic form) is a function that selects for each game in strategic form G a (possibly empty) set of its outcomes as the solutions of G.

Classical Game-Theoretic Approach:
Solution rules are formulated with the understanding that:
(a) the players have *mutual knowledge of unbounded order of the game* (i.e. all players know the game and know that all know the game and so on...$\rightarrow$ *common knowledge of the game)*
(b) the players are taken to be familiar with the game, hence any solution rule is meant to predict the behaviour of competent players, and *should apply even to a single play of the game*;
(c) the population of agents playing the game is invariably *one* and it is precisely *the set of players themselves*.

Any departure from one of the former assumptions and understandings amounts to venturing into the realm of the *evolutionary game-theoretic approach.*

Non-cooperative versus *Cooperative* solution rules

Non-cooperative solution rules are designed to apply to interactive situations where *coalition formation is a negligible event* (e.g. communication among players is very costly or most unlikely).

Cooperative solution rules are designed to apply to interactive situations where *coalition formation is a common event* (e.g. players have access to cheap communication channels).

The demarcation between cooperative and non-cooperative solution rules, however, is not necessarily neat and crisp...

Basic Non-cooperative Solution Rules for Games in Strategic Form

DEFINITION **Nash Equilibrium (NE) of a game in strategic form**

$$G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N}):$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i \in N} S_i$ such that

for every $i \in N$ and $t_i \in S_i$, $h(s) \succcurlyeq_i h(t_i, s_{-i})$

(in words, a NE of G is a profile of strategies that are *mutually best replies to the other strategies of the profile itself,*

or *a profile of strategies without individual objections* where an *individual objection* to strategy profile $s = (s_1, ..., s_n)$ in G

is a pair $(i, t_i) \in N \times S_i$ such that $h(t_i, s_{-i}) \succ_i h(s)$ i.e.

$[h(t_i, s_{-i}) \succcurlyeq_i h(s)$ and *not* $h(s) \succcurlyeq_i h(t_i, s_{-i})]$).

Nash Equilibrium Existence Theorem (Nash (1950)).
Let $G = (N, X, (S_i)_{i\in N}, h, (\succcurlyeq_i)_{i\in N})$ be a game in strategic form such that, for every $i \in N$:
(i) there exists a positive integer k such that $S_i \subseteq \mathbb{R}^k$, and S_i is both *compact* and *convex*;
(ii) $\succcurlyeq_i$is a *total preorder;*
(iii) $\succcurlyeq_i$is *continuous* i.e. for each $x \in X$ both the *upper contour* $UC(\succcurlyeq_i, x) = \{y \in X : y \succcurlyeq_i x\}$
and the *lower contour* $LC(\succcurlyeq_i, x) = \{y \in X : x \succcurlyeq_i y\}$ are *closed sets*;
(iv) for all $s = (s_1, ..., s_n) \in \prod_{i\in N} S_i$, $\{t_i \in S_i : h(t_i, s_{-i}) \succcurlyeq_i h(s)\}$ is *convex.*
(v) h is a *continuous* function.
Then, G has a Nash equilibrium namely $NE(G) \neq \varnothing$.

REMARK The proof of this theorem relies on anyone of two (equivalent) fixed point theorems: the *Brouwer fixed point theorem* for continuous functions on convex compact Euclidean sets, and the *Kakutani fixed point theorem* for correspondences with closed graph and non-empty compact and convex values on convex compact Euclidean sets.

Kakutani Fixed Point Theorem. Let $K \subseteq \mathbb{R}^k$ be nonempty compact and convex and $F : K \to\to K$ a *correspondence* which satisfies the following conditions: $F(x)$ is nonempty, compact and convex for every $x \in K$, and F has a *closed graph* (namely for any pair of convergent sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in K with $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$, if $y_n \in F(x_n)$ for each n then $y^* \in F(x^*)$). Then F has a *fixed point*, namely there exists $x \in K$ such that $x \in F(x)$.

Sketch of proof. Let us define the individual best reply correspondences of G

$$B_i : \prod_{i \in N} S_i \rightarrow\rightarrow S_i \text{ for each } i \in N : \text{ for each } s \in \prod_{i \in N} S_i$$

$B_i(s) = \{s_i' \in S_i : h(s_i', s_{-i}) \succcurlyeq_i h(t_i, s_{-i}) \text{ for all } t_i \in S_i\}$

and the best reply correspondence of G

$$B^G : \prod_{i \in N} S_i \rightarrow\rightarrow \prod_{i \in N} S_i,$$

$$\text{where } B^G(s) = \prod_{i \in N} B_i(s).$$

Observe that (i) s is a Nash equilibrium of G iff it is a fixed point of B^G

(Sketch of proof, continued)

Moreover, each $B_i(s)$ is

(ii) nonempty and compact by the Weierstrass theorem because $\succcurlyeq_i$ is representable by a continuous utility function

(iii) convex by local convexity of $\succcurlyeq_i$

(iv) has a closed graph because of continuity of $\succcurlyeq_i$ and h.

Hence B^G also inherits properties (ii) (iii) (iv) by construction.

Thus, Kakutani Theorem applies and B^G has a fixed point.

EXAMPLE **The mixed extension of a finite game in strategic form** $G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N})$ (i.e. S_i is *finite* for every $i \in N$)

DEFINITION (**Lotteries on a set)** A set of lotteries on a set Y is a set L_Y endowed with a family $\{\oplus_\lambda : \lambda \in [0,1] \subseteq \mathbb{R}\}$ of binary *combination operations* which satisfy the following conditions (where $\oplus_\lambda(a,b) := \lambda a \oplus (1-\lambda)b$))

(i) for any $a, b \in L_Y$, $1a \oplus 0b = a$;

(ii) for any $a, b \in L_Y$, $\lambda a \oplus (1-\lambda)b = (1-\lambda)b \oplus \lambda a$;

(iii) for any $a, b \in L_Y$ and $\lambda, \mu \in [0,1]$,
$\lambda[\mu a \oplus (1-\mu)b] \oplus (1-\lambda)b = \lambda\mu a \oplus (1-\lambda\mu)b$.

Theorem (Von Neumann 1947): Let L_X be a set of lotteries on X, and $\succcurlyeq$ a binary preference relation on L_X that satisfies the following conditions:
(i) (Total Preordering) $\succcurlyeq$ is a total preorder;
(ii) (Independence) for every $a, b, c \in L_X$ and $\lambda \in [0, 1]$, if $a \succcurlyeq b$ then $\lambda a \oplus (1-\lambda)c \succcurlyeq \lambda b \oplus (1-\lambda)c$;
(iii) (Archimedean Property) for every $a, b, c \in L_X$ if $a \succ b \succ c$ then there exists $\lambda \in [0, 1]$ such that
$b \sim \lambda a \oplus (1-\lambda)c$ (namely, both $b \succcurlyeq \lambda a \oplus (1-\lambda)c$ and $\lambda a \oplus (1-\lambda)c \succcurlyeq b$ hold).
Then $\succcurlyeq$ is representable by an *expected utility (EU) function* u, namely a function $u : L_X \to \mathbb{R}$ such that
for every $a, b, c \in L_X$, and $\lambda \in [0, 1]$, $u(a) \geq u(b)$ if and only if $a \succcurlyeq b$, and
if $c = \lambda a \oplus (1-\lambda)b$ then $u(c) = \lambda u(a) + (1-\lambda)u(b)$.
Moreover, u is an *interval scale* i.e. is uniquely defined up to isotonic affine transformations
(v is an EU representation of $\succcurlyeq$ iff $v(.) := \alpha u(.) + \beta$ with $\alpha, \beta \in \mathbb{R}$, and $\alpha > 0$).

DEFINITION **The mixed extension of a finite game in strategic form** $G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N})$ (i.e. S_i is *finite* for every $i \in N$)

is the game in strategic form defined as follows

$\widetilde{G} = (N, L_X, (\widetilde{S}_i)_{i \in N}, \widetilde{h}, (\widetilde{\succcurlyeq}_i)_{i \in N})$

where

L_X is a set of lotteries on X (observe that $X \subseteq L_X$)

$\widetilde{S}_i$ a set of lotteries on S_i, for each $i \in N$

$\widetilde{h} : \prod_{i \in N} \widetilde{S}_i \to L_X$ is the extension of h to $\prod_{i \in N} \widetilde{S}_i$

$\widetilde{\succcurlyeq}_i \subseteq L_X \times L_X$ is an extension of $\succcurlyeq_i$that satisfies

Total Preordering, Independence, Archimedean Property

and thus

has an expected utility representation u_i, for each $i \in N$

EXAMPLE: Computing the Nash Equilibria of Stag Hunt

Take a STAG HUNT with its mixed extension defined as follows

1/2	s_2	h_2
s_1	a	b
h_1	c	d

1/2	s_2	h_2
s_1	9; 9	0; 8
h_1	8; 0	7; 7

$\succcurlyeq_1 := (acdb) \longrightarrow u_1 \uparrow$ (rows)

$\succcurlyeq_2 := (abdc) \longrightarrow u_2 \uparrow$ (columns)

Warning: u_1 and u_2 are now *expected utility functions* (not just ordinal utility functions).

Pure Nash Equilibria: The profiles of mutually best replies in pure strategies (the original ones, i.e. degenerate lotteries) are easily found: they are of course (s_1, s_2) and (h_1, h_2).
(continues)

(continues)
(Properly) Mixed Nash Equilibria: In order to compute them we rely on the following
Claim: A properly mixed strategy σ_i of player i is a best reply to a given strategy σ_j of the other player j only if any two pure strategies which are assigned a positive probability mass according to σ_i do ensure the same expected utility in terms of u_i when player j plays the given strategy σ_j.

Thus, posit $x :=$probability of playing s_1 in a mixed strategy
$y :=$probability of playing s_2 in a mixed strategy

and look at the game

1/2	s_2	h_2
s_1	$9;9$	$0;8$
h_1	$8;0$	$7;7$

The relevant equations are:
concerning player 1: $9y + 0(1-y) = 8y + 7(1-y)$ or $8y = 7$
concerning player 2: $9x + 0(1-x) = 8x + 7(1-x)$ or $8x = 7$
hence the mixed NE is $\left(\frac{7}{8}s_1 \oplus \frac{1}{8}h_1, \frac{7}{8}s_2 \oplus \frac{1}{8}h_2\right)$.

REFINEMENTS OF NASH EQUILIBRIUM

DEFINITION **Strict Nash Equilibrium of a game in strategic form**

$$G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N}) :$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i \in N} S_i$ such that

for every $i \in N$ and $t_i \in S_i \setminus \{s_i\}$, $h(s) \succ_i h(t_i, s_{-i})$

(in words, a strict NE of G is a profile of strategies that are *mutually* **unique** *best replies to the other strategies of the profile itself)*

REMARK Let us make the common place assumption that less costly mistakes are definitely more likely than more costly ones. Accordingly, inconsequential mistakes are the most common sort of mistake. In that connection, since a strict NE by definition does not admit any inconsequential mistake, it follows that it can be regarded as *a NE that is robust against the most common sort of mistake in executing the planned strategies.*

DEFINITION **Dominant Strategy Equilibrium (DSE) of a game in strategic form**

$$G = (N, X, (S_i)_{i\in N}, h, (\succcurlyeq_i)_{i\in N}) :$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i\in N} S_i$ such that

for every $i \in N$ and $t \in \prod_{i\in N} S_i$, $h(s_i, t_{-i}) \succcurlyeq_i h(t)$.

REMARK. A DSE is a NE that is robust against any error of prediction concerning the behaviour of other players.

A WEAKENING OF NASH EQUILIBRIUM

DEFINITION **Minimal Sets Closed Under Rational Behaviour (Minimal CURB sets) of a game in strategic form**

$$G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N}) :$$

a set $S' := \prod_{i \in N} S'_i \subseteq \prod_{i \in N} S_i$ of strategy-profiles of G such that

(i) for any $s = (s_1, ..., s_n) \in S'$, $i \in N$, $t_i \in S_i$:

if $h(t_i, s_{-i}) \succcurlyeq_i h(s'_i, s_{-i})$ for every $s'_i \in S'_i$ then $t_i \in S'_i$

(ii) there is no $S'' := \prod_{i \in N} S''_i \subset S'$ that satisfies condition (i) above.

ALTERNATIVE NON-COOPERATIVE SOLUTION RULES
(1) RATIONALIZABLE STRATEGY-PROFILES

DEFINITION **Rationalizable Strategy-Profiles (RSP) of a game in strategic form**

$$G = (N, X, (S_i)_{i\in N}, h, (\succcurlyeq_i)_{i\in N}) :$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i\in N} S_i \in S^K$

where $S^K := S^{K-1}$,

$$S^0 := \prod_{i\in N} S_i^0 = \prod_{i\in N} S_i \ ,$$

and for all $T = 1, ..., K-1$

$$S^T := \left\{ \begin{array}{c} (s_1, ..., s_n) \in \prod_{i\in N} S_i^{T-1} : \text{for each } i \in N \\ \text{there is no } s_i' \in S_i^{T-1} \text{ such that} \\ h(s_i', t_{-i}) \succ_i h(s_i, t_{-i}) \text{ for every } t_{-i} \in \prod_{j\in N, j\neq i} S_j^{T-1} \end{array} \right\},$$

and $S^T \subset S^{T-1}$.

In words, a strategy-profile s of G is *rationalizable* if and only if it is one of those profiles that survive *the iterated elimination of strictly dominated strategies.*

REMARK: It is easily checked that RSP selects *one* strategy profile of PD, APD, WL, IH, HC and *all the strategy-profiles of the other* 2×2 games introduced above. RSP also selects a unique strategy-profile of the 'competition by differentiation' game as defined previously: notice that the RSP-outcome of this game is weakly (Pareto-)efficient but *not* (Pareto-)efficient.

(2) ITERATIVELY NON-DOMINATED STRATEGY-PROFILES AND DOMINANCE SOLVABILITY

DEFINITION **Iteratively Non-Dominated Strategy-Profiles of a game in strategic form**

$$G = (N, X, (S_i)_{i\in N}, h, (\succsim_i)_{i\in N}):$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i\in N} S_i \in S^K$

where $S^K := S^{K-1}$,

$$S^0 := \prod_{i\in N} S_i^0 = \prod_{i\in N} S_i \ ,$$

and for all $T = 1, ..., K-1$

$$S^T := \left\{ \begin{array}{c} (s_1, ..., s_n) \in \prod_{i\in N} S_i^{T-1} : \text{for each } i \in N \\ \text{there is no } s_i' \in S_i^{T-1} \text{ such that} \\ h(s_i', t_{-i}) \succsim_i h(s_i, t_{-i}) \text{ for all } t_{-i} \in \prod_{j\in N, j\neq i} S_j^{T-1}, \text{ and} \\ h(s_i', t_{-i}) \succ_i h(s_i, t_{-i}) \text{ for some } t_{-i} \in \prod_{j\in N, j\neq i} S_j^{T-1} \end{array} \right\},$$

and $S^T \subset S^{T-1}$.

DEFINITION **Sophisticated Equilibria and Dominance Solvability**:
If the set S^K of iteratively non-dominated strategy-profiles of a game in strategic form $G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N})$ is such that for every $i \in N$, and every $s, s' \in S^K$, $h(s) \sim_i h(s')$ holds, then the strategy-profiles of S^K are said to be **sophisticated equilibria** of G, and G itself is said to be **dominance solvable**.

EXAMPLE Consider the **Guess the Average prize-contest game** $G^{GA} = (N, X = \{0,1\}^N, (S_i = [1, ..., 999])_{i \in N}, h^{W(2/3)}, (\succcurlyeq_i)_{i \in N})$ as previously defined. It can be quite easily checked that G^{GA} is indeed *dominance solvable* and its (unique) *sophisticated equilibrium* is $s := (s_1 = 1, ..., s_n = 1)$. (Check it!)

Two Cooperative Extensions of Nash Equilibrium

DEFINITION **Strong (Nash) Equilibrium (SE) of a game in strategic form**

$$G = (N, X, (S_i)_{i \in N}, h, (\succsim_i)_{i \in N}) :$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i \in N} S_i$ such that

for every $C \subseteq N$ and $t_C \in \prod_{i \in C} S_i$, there exists $i \in C$ such that

$h(s) \succsim_i h(t_C, s_{N \setminus CN})$

(in words, a SE of G is *a profile of strategies without coalitional objections* where a *coalitional objection* to strategy profile $s = (s_1, ..., s_n)$ in G is a pair $(C, t_C) \in 2^N \times \prod_{i \in C} S_i$ such that $h(t_C, s_{N \setminus C}) \succ_i h(s)$ i.e. $[h(t_C, s_{N \setminus C}) \succsim_i h(s)$ and *not* $h(s) \succsim_i h(t_C, s_{N \setminus C})]$) for each $i \in C$.

DEFINITION **Coalition-Proof Nash Equilibrium (CPE) of a game in strategic form**

$$G = (N, X, (S_i)_{i \in N}, h, (\succsim_i)_{i \in N}):$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i \in N} S_i$ such that

for every $C \subseteq N$ and $t_C \in \prod_{i \in C} S_i$, such that $h(s) \succ_i h(t_C, s_{N \setminus CN})$ there exists $C' \subseteq C$ and $t'_{C'}$ such $h(t'_{C'}, t_{C \setminus C'}, s_{N \setminus CN}) \succ_i h(t_C, s_{N \setminus CN})$ for each $i \in C'$

(in words, a CPE of G is *a profile of strategies such that any coalitional objection* to that profile has an *internal objection from within the same coalition)*.

Coalitional equilibrium and the core

DEFINITION **Coalitional Equilibrium (CoalE) of a game in strategic form**

$$G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N}) :$$

a strategy profile $s = (s_1, ..., s_n) \in \prod_{i \in N} S_i$ such that

for every $C \subseteq N$ and $t_C \in \prod_{i \in C} S_i$, if $h(t_C, s_{-i}) \succ_i h(s)$ for all $i \in C$

then there exist $D \subseteq N \setminus C$ and $t_D \in \prod_{i \in D} S_i$ such that

$[h(t_D, t_C, s_{-C \cup D}) \succ_i h(s)$ for all $i \in D$ and $h(s) \succcurlyeq_j h(t_D, t_C, s_{-C \cup D})$ for some $j \in C]$

(in words, a coalitional equilibrium of G is *a profile of strategies such that every coalitional objection* to it has a *coalitional counter-objection*).

REMARK. The outcomes of coalitional equilibria are the **core-outcomes** of G. Observe that SE-outcomes are a subset of core-outcomes.

Von Neumann-Morgenstern (VNM) Stable Sets

DEFINITION **VNM Stable Set of a game in strategic form**
$$G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N}) :$$
a set $\mathcal{S}$ of strategy profiles of G such that
(i) (*Internal Stability*): for every $s \in \mathcal{S}$, and every $C \subseteq N$ and $t_C \in \prod_{i \in C} S_i$ with $(t_C, s_{-C}) \in \mathcal{S}$, if $h(t_C, s_{-C}) \succ_i h(s)$ for all $i \in C$ then there exist $D \subseteq N \setminus C$ and $t_D \in \prod_{i \in D} S_i$ such that $[(t_D, t_C, s_{-C \cup D}) \in \mathcal{S}$, $h(t_D, t_C, s_{-C \cup D}) \succ_i h(t_C, s_{-C})$ for all $i \in D$, and $h(s) \succcurlyeq_j h(t_D, t_C, s_{-C \cup D})$ for some $j \in C]$.
(ii) (*External Stability*): for every $s \in \prod_{i \in N} S_i \setminus \mathcal{S}$, there exist $s' \in \mathcal{S}$, $C \subseteq N$ and $t_C \in \prod_{i \in C} S_i$ with $s' = (t_C, s_{-C})$ such that $h(t_C, s_{-C}) \succ_i h(s)$ for all $i \in C$, and there exist no $D \subseteq N \setminus C$ and $t_D \in \prod_{i \in D} S_i$ such that $[h(t_D, t_C, s_{-C \cup D}) \succ_i h(s)$ for all $i \in D$ and $h(s) \succcurlyeq_j h(t_D, t_C, s_{-C \cup D})$ for some $j \in C]$.

REMARK

In words, a set $\mathcal{S}$ of strategy-profiles of G is VNM-stable if each of its profiles can rely on coalitional counter-objections against any coalitional objection belonging to $\mathcal{S}$ itself, and any strategy-profile *not* in $\mathcal{S}$ is vulnerable to a coalitional objection belonging to $\mathcal{S}$ which does not admit any coalitional counterobjection). Thus, a VNM stable set can also be regarded as a *stable standard of behaviour within the given game*, hence a *precise game-theoretical model of an* **institution**.

REMARK

It should be emphasized the contrast between two basic versions of *counterobjections: counterobjections as external counterobjections (EC)*, and *counterobjections as internal counterobjections (IC).* Notice that the counterobjections involved in the definitions of VNM stable sets and of coalitional equilibria are of EC, while those involved in the definition of coalition-proof Nash equilibria are IC. The difference between the respective predictions is remarkable. (Check that claim on PD and other 2×2 bimatrix games!).

Correlated Equilibrium (CE)

DEFINITION **Correlated Equilibrium (CE) of a (finite) game in strategic form** $G = (N, X, (S_i)_{i\in N}, h, (\succcurlyeq_i)_{i\in N})$ with mixed extension $\widetilde{G} = (N, L_X, (\widetilde{S}_i)_{i\in N}, \widetilde{h}, (\widetilde{\succcurlyeq}_i)_{i\in N})$:

an array $\mathcal{C}(G) = ((\Omega, \pi), (P_i)_{i\in N}, (\sigma_i)_{i\in N})$ where

- (Ω, π) is a *finite probability space*, namely Ω is a finite set of possible states and π a probability measure on 2^{Ω} (or equivalently a probability distribution -or lottery- on Ω);
- $(P_i)_{i\in N}$ is a profile of *partitions* P_i of Ω (the *information partition* of player i, for any $i \in N$);
- a profile $(\sigma_1, ..., \sigma_n)$ of *conditional strategies* i.e. of P_i-measurable functions $\sigma_i : \Omega \to S_i$ (namely, such that $\sigma_i(\omega) = \sigma_i(\omega')$ for any ω,$\omega' \in \Omega$ which belong to a same block P_{ij} of partition P_i), $i \in N$,

and for each player $i \in N$ and every P_i-measurable function $\tau_i : \Omega \to S_i$,

$$\sum_{\omega\in\Omega} \pi(\omega) \cdot \widetilde{u}_i(h(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega\in\Omega} \pi(\omega) \cdot \widetilde{u}_i(h(\tau_i(\omega), \sigma_{-i}(\omega))$$

(where $\widetilde{u}_i : L_X \to \mathbb{R}$ denotes an EU function which represents $\widetilde{\succcurlyeq}_i$).

Special case

DEFINITION **Correlated Equilibrium (CE) of a (finite) game in strategic form** $G = (N, S \equiv \prod_{i \in N} S_i, (S_i)_{i\in N}, id, (\succcurlyeq_i)_{i \in N})$ with mixed extension $\widetilde{G} = (N, L_S, (\widetilde{S}_i)_{i \in N}, \widetilde{h}, (\widetilde{\succcurlyeq}_i)_{i\in N})$:

an array $\mathcal{C}(G) = ((S, \pi), (P_i)_{i\in N}, (\sigma_i)_{i \in N})$ where

- (S, π) is a *finite probability space*, namely π a probability measure on 2^S (or equivalently a probability distribution -or lottery- on S);
- $(P_i)_{i \in N}$ is a profile of *partitions* $P_i = \{\{t \in S : t_i = s_i\} : s_i \in S_i\}$ of S (the *information partition* of player i, for any $i \in N$);
- a profile $(\sigma_1, ..., \sigma_n)$ of *conditional strategies* i.e. of P_i-measurable functions $\sigma_i : S \to S_i$ (namely, such that $\sigma_i(s) = \sigma_i(s')$ for any pair of strategy-profiles s,$s' \in S$ with $s_i = s'_i$), $i \in N$, and for each player $i \in N$ and every P_i-measurable function $\tau_i : S \to S_i$,

$$\sum_{s \in S} \pi(s) \cdot \widetilde{u}_i(h(\sigma_i(s), \sigma_{-i}(s))) \geq \sum_{s\in S} \pi(s) \cdot \widetilde{u}_i(h(\tau_i(s), \sigma_{-i}(s)))$$

(where $\widetilde{u}_i : L_S \to \mathbb{R}$ denotes an EU function which represents $\widetilde{\succcurlyeq}_i$).

A classical interpretation/implementation of Correlated Equilibrium.
One *public lottery* is run and an outcome-dependent (and typically *private*) signal/recommendation is sent to each player. A correlated equilibrium is a conditional-strategy-profile such that each player -knowing the lottery- has never an incentive to reject the recommendations she receives.

REMARK
Mixed Nash Equilibria as a particular 'degenerate' case of Correlated Equilibria with *stochastically independent* choice of strategies (or zero-correlation of strategy-choices). The interpretation proposed above clearly does *not* apply to this 'degenerate' case which rather amounts to running *several private lotteries*, one for each player.

EXAMPLE **Traffic lights as a correlated equilibrium generating device**
Consider the following version of the mixed extension of **Hawk-Dove (HD),** sometimes also named **Crossing Game**

1/2	s_2	g_2
s_1	2; 2	1; 3
g_1	3; 1	0; 0

,

where s_i and g_i, $i = 1, 2$, denote 'stop' and 'go', respectively, and the utility values are of course *expected utility values.*

It is easily checked that a correlated equilibrium of this game is the array $((S, \pi), (P_i)_{i \in N}, (\sigma_i)_{i \in N})$ with
$\pi((s_1, g_2)) = \pi((g_1, s_2)) = \frac{1}{2}$,
$P_i = \{\{u \in (\{s_1, g_1\} \times \{s_2, g_2\} : u_i = t_i\} : t_i \in \{s_i, g_i\}\}$
and for every $t \in \{s_1, g_1\} \times \{s_2, g_2\}$, $\sigma_i(t) = t_i$, $i = 1, 2$.
That correlated equilibrium results in the following lottery on strategy profiles:

$$\frac{1}{2}(s_1, g_2) \oplus \frac{1}{2}(g_1, s_2).$$

Notice that this is precisely the standard behaviour induced by the introduction of *traffic lights* at crossroads.

Coarse Correlated Equilibrium

DEFINITION **Coarse Correlated Equilibrium (CCE) of a (finite) game in strategic form** $G = (N, X, (S_i)_{i \in N}, h, (\succcurlyeq_i)_{i \in N})$ with mixed extension $\widetilde{G} = (N, L_X, (\widetilde{S}_i)_{i \in N}, \widetilde{h}, (\widetilde{\succcurlyeq}_i)_{i \in N})$:

an array $\mathcal{C}(G) = ((\Omega, \pi), (P_i)_{i \in N}, (\sigma_i)_{i \in N})$ where

- (Ω, π) is a *finite probability space*, namely Ω is a finite set of possible states and π a probability measure on 2^{Ω} (or equivalently a probability distribution -or lottery- on Ω);
- $(P_i)_{i \in N}$ is a profile of *partitions* P_i of Ω (the *information partition* of player i, for any $i \in N$);
- a profile $(\sigma_1, ..., \sigma_n)$ of *conditional strategies* i.e. of P_i-measurable functions $\sigma_i : \Omega \to S_i$ (namely, such that $\sigma_i(\omega) = \sigma_i(\omega')$ for any ω,$\omega' \in \Omega$ which belong to a same block P_{ij} of partition P_i), $i \in N$,

and for each player $i \in N$ and every **constant** function $\tau_i : \Omega \to S_i$,

$$\sum_{\omega \in \Omega} \pi(\omega) \cdot \widetilde{u}_i(h(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) \cdot \widetilde{u}_i(h(\tau_i(\omega), \sigma_{-i}(\omega))$$

(where $\widetilde{u}_i : L_X \to \mathbb{R}$ denotes an EU function which represents $\widetilde{\succcurlyeq}_i$).

A classical interpretation/implementation of Coarse Correlated Equilibrium

A public lottery is run and an outcome-dependent private signal/recommendation is sent to each player. A correlated equilibrium conditional-strategy-profile is such that each player -knowing the lottery- prefers to *precommit to following the recommendations* to be received from her/his private signals as determined by the outcomes of the lottery, rather than choosing a fixed *constant* strategy.

REMARK

(i) A special version of CCE with $\Omega = \prod_{i \in N} S_i$ is also available along the same lines of the CE case previously considered.

(ii) A constant function $\tau_i : \Omega \to S_i$ is obviously also P_i-measurable, and amounts to a fixed strategy which does not depend of the information state of the player. Hence the *conditional strategies* selected by players at a CCE are only required to be regarded as better than any alternative *fixed* strategy. In other terms, players are *not* allowed to adjust their choices after receiving their private information concerning the outcome of the public lottery π. They can only choose between any fixed strategy and commitment to a certain pre-fixed way to react to the private signals to be received from the outcomes of the public lottery π as coded in a conditional strategy chosen *before* running the public lottery.

EXAMPLE: Coarse Correlated Equilibrium improves on Correlated Equilibria in the 'Competition by Differentiation' game

Consider again the '**Competition by differentiation**' bimatrix game :

1/2	L	M	H
L	$0;0$	$0;2$	$1;3$
M	$2;0$	$1;1$	$2;0$
H	$3;1$	$0;2$	$0;0$

It is easily checked that the following lottery on strategy-profiles of this game, namely

$$L_S := \frac{1}{2}(L,H) \oplus \frac{1}{2}(H,L)$$

is the conditional strategy-profile of a CCE, with expected utility/payoffs $(2,2)$, a strict Pareto-improvement with respect to $(1,1)$ (the expected utility/payoff induced by (M,M), which is indeed the unique rationalizable strategy-profile but also the unique NE and the unique CE of that game). [Check it all!]

EVOLUTIONARILY STABLE STRATEGIES IN SYMMETRIC STRATEGIC GAMES: A VERY SHORT FORAY INTO EVOLUTIONARY GAME THEORY

Evolutionary game theory: moving the focus *from players to strategies* regarded as *cultural replicators* (or '*memes*'): evolutionarily stable strategies as strategies that once established as prevailing strategies within a population of agents are immune to *single, local* mutations.

DEFINITION **Symmetric Games in Strategic Form:** A 2-player game in strategic form $G = (\{1,2\}, X, (S_i)_{i=1,2}, h, (\succcurlyeq_i)_{i=1,2})$ is **symmetric** if there exists a labeling of strategies such that $S_1 = S_2 = \overline{S}$ and for any $s, t \in \overline{S}$, and $i, j \in \{1,2\}$ with $i \neq j$, $h(s,t) \succcurlyeq_i h(t,s)$ entails $h(t,s) \succcurlyeq_j h(s,t)$.

DEFINITION **Evolutionarily Stable Strategy (ESS)** Let $G = (\{1,2\}, X, (S_i)_{i=1,2}, h, (\succcurlyeq_i)_{i=1,2})$ be a symmetric game in strategic form. A strategy $s \in \overline{S} = S_1 = S_2$ is an **evolutionarily stable strategy (ESS)** of G if and only if, for any $i \in \{1,2\}$ and $t \in \overline{S} \setminus \{s\}$:
(i) $h(s,s) \succcurlyeq_i h(t,s)$, and (ii) if $h(s,s) \sim_i h(t,s)$ then $h(s,t) \succ_i h(t,t)$.

CLAIM
(1) If $s \in S_1 = S_2$ is an ESS of G then (s,s) is a (symmetric) Nash equilibrium of G.
(2) If (s,s) is a (symmetric) *strict* Nash equilibrium of G then s is an ESS of G. (Check (1) and (2)!)

Representation Theorem for Evolutionarily Stable Strategies (Maynard Smith (1974)):
Let $G = (\{1,2\}, X, (S_i)_{i=1,2}, h, (\succcurlyeq_i)_{i=1,2})$ be the mixed extension of a finite symmetric game in strategic form, and u an expected utility function that represents $\succcurlyeq_1$. Then, $s \in S_1 = S_2 = \overline{S}$ is an ESS of G if and only if for each $t \in \overline{S}$ there exists an $\epsilon_t \in (0,1)$ such that for every $\epsilon \leq \epsilon_t$:
$(*) \qquad (1-\epsilon) \cdot u(s,s) + \epsilon \cdot u(s,t) > (1-\epsilon) \cdot u(t,s) + \epsilon \cdot u(t,t).$

REMARK In words, whenever s is established as the typical strategy of the relevant population of agents, s enjoys an *invasion barrier* ϵ_t against any possible *single mutant* strategy t.

Proof $\Longrightarrow$ suppose $s \in \overline{S}$ is such that for each $t \in \overline{S} \quad u(s,s) \geq u(t,s)$ and

$$u(s,s) = u(t,s) \text{ entails } u(s,t) > u(t,t).$$

Then, take $\Delta := u(s,s) - u(t,s)$ and $\Gamma := u(s,t) - u(t,t)$.
Clearly, the current thesis $(*)$ may be rewritten as follows
$(1-\epsilon)\cdot\Delta + \epsilon\cdot\Gamma > 0$.
If $\Delta = 0$ then by hypothesis $\Gamma > 0$, whence
$(1-\epsilon)\cdot\Delta + \epsilon\cdot\Gamma = \epsilon\cdot\Gamma > 0$ for each $\epsilon \in (0,1)$.
If $\Delta > 0$ two cases are to be distinguished:
suppose first that $\Gamma \geq 0$: then for each $\epsilon \in (0,1)$,
both $\epsilon\cdot\Gamma \geq 0$ and $(1-\epsilon)\cdot\Delta > 0$, hence the thesis holds.
Next, suppose that $\Gamma < 0$: then $(1-\epsilon)\cdot\Delta > -\epsilon\cdot\Gamma$ iff
$(1-\epsilon)\cdot\Delta > \epsilon\cdot|\Gamma|$ iff $\epsilon < \frac{\Delta}{\Delta+|\Gamma|}$.

Proof (cont.)

$\Longleftarrow$ suppose now that $(*)$ holds i.e. equivalently with the previous notation

$$(1-\epsilon)\cdot\Delta+\epsilon\cdot\Gamma>0 \text{ for every } \epsilon\in(0,\epsilon_t)$$

Now suppose that $\Delta<0$: then $(1-\epsilon)\cdot\Delta+\epsilon\cdot\Gamma<0$ for every $\epsilon<\frac{\Delta}{|\Delta|+\Gamma}$, contradicting $(*)$.

Hence $\Delta\geq 0$, thus $u(s,s)\geq u(t,s)$. Now, suppose that $u(s,s)=u(t,s)$, namely $\Delta=0$.

Then, $(1-\epsilon)\cdot\Delta+\epsilon\cdot\Gamma>0$ iff $\epsilon\cdot\Gamma>0$ iff $\Gamma>0$ iff $u(s,t)>u(t,t)$ and the thesis follows.

EXAMPLE **A symmetric game with no ESS**

Consider the mixed extension $\widetilde{G}$ of the following bimatrix game

1/2	S	T	U
S	$\gamma;\gamma$	$1;-1$	$-1;1$
T	$-1;1$	$\gamma;\gamma$	$1;-1$
U	$1;-1$	$-1;1$	$\gamma;\gamma$

with $\gamma \in [0,1] \subseteq \mathbb{R}$. It can be shown that $\widetilde{G}$ has no ESS (Prove it!).

A **Game in Coalitional Form** is a structure $G = (N, X, E, (t_i(\succcurlyeq_i))_{i \in N})$ where

N set of *players*, $N \neq \varnothing$ and usually finite ($N = \{1, ..., n\}$)

X set/space of *outcomes*

$E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ *effectivity function* with the following 'boundary' properties:

(i) *Adequacy (AD):* for all $S \subseteq N$, $S \neq \varnothing$, $X \in E(S)$

(ii) *Consistency (C):* for all $S \subseteq N$, $\varnothing \notin E(S)$

(iii) *Non-triviality (N):* for all $Y \subset X$, $Y \notin E(\varnothing)$

(iv) *Double Normalization (DN):* $E(\varnothing) \subseteq \{X\}$

(where $\mathcal{P}(N) := \{S : S \subseteq N\}$ denotes the set of *coalitions,* $\mathcal{P}(X) := \{Y : Y \subseteq X\}$ the set of *events*, $\mathcal{P}(\mathcal{P}(X)) := \{\mathcal{X} : \mathcal{X} \subseteq \mathcal{P}(X)\}$ the set of *collections of events*)

$t_i(\succcurlyeq_i)$ *type* of player i, $i \in N$, including his/her *preference relation* $\succcurlyeq_i \subseteq X \times X$

(usually taken to be a *total preorder* i.e. -for each $x, y, z \in X$, *reflexive:* $x \succcurlyeq_i x$, *transitive: if* $x \succcurlyeq_i y$ and $y \succcurlyeq_i z$ then $x \succcurlyeq_i z$ and *connected:* if $x \neq y$ then $x \succcurlyeq_i y$ or $y \succcurlyeq_i x$ - and in that case representable by an *utility function* $u_i : X \to \mathbb{R}$).

REMARK The effectivity function is the main component of the *coalitional game form* $[G] = (N, X, E)$. It describes the *decision power of coalitions* by listing the entire collection of events each coalition is able of enforce independently of the behaviour of other players (the rules of the coalitional game or 'Who Can Do What').
Observe that the effectivity function E can also be written as $E : 2^N \rightarrow 2^{2^X}$ since for any set A, $\mathcal{P}(A) \simeq 2^A$. Indeed, E is a *boolean function* (a mapping between boolean algebras).

A LIST OF BASIC KEY PROPERTIES TO CLASSIFY EFFECTIVITY FUNCTIONS

Sovereignty (SV): for all $\varnothing \neq Y \subseteq X$ there exists $S \subseteq N$ such that $Y \in E(S)$

X-Monotonicity (X-MON): for all $Y, Z \subseteq X$ and $S \subseteq N$, if $Y \in E(S)$ and $Y \subseteq Z$ then $Z \in E(S)$

N-Monotonicity (N-MON): for all $Y \subseteq X$ and $S, T \subseteq N$, if $Y \in E(S)$ and $S \subseteq T$ then $Y \in E(T)$

Monotonicity (MON): E satisfies both X-MON and N-MON

Regularity (REG): for all $S \subseteq N$, $S \neq \varnothing$ and all $Y, Z \subseteq X$, if $Y \in E(S)$ and $Z \in E(N \setminus S)$ then $Y \cap Z \neq \varnothing$

Maximality (MAX): for all $S \subseteq N$, $S \neq \varnothing$ and all $Y \subseteq X$, if $Y \notin E(S)$ then there exists $Z \subseteq X$ such that $Z \in E(N \setminus S)$ and $Y \cap Z = \varnothing$

Superadditivity (SUPA): for all $S, T \subseteq N$, $Y, Z \subseteq X$, if $Y \in E(S)$, $Z \in E(T)$ and $S \cap T = \varnothing$ then $Y \cap Z \in E(S \cup T)$

Convexity (CONV): for all $S, T \subseteq N$, $Y, Z \subseteq X$, if $Y \in E(S)$ and $Z \in E(T)$ then either $Y \cap Z \in E(S \cup T)$ or $Y \cup Z \in E(S \cap T)$

[continues]

[continues]
Anonymity (AN): for all $S, T \subseteq N$, and $Y \subseteq X$ if $Y \in E(S)$ and $|S| = |T|$ then $Y \in E(T)$
Neutrality (NT): for all $S \subseteq N$, and $Y, Z \subseteq X$ if $Y \in E(S)$ and $|Y| = |Z|$ then $Z \in E(S)$
Simplicity (SIMP): there exists $\mathcal{W} \subseteq \mathcal{P}(N)$ such that for all $S \subseteq N$, $S \neq \varnothing$ and $\varnothing \neq Y \subseteq X$, $Y \in E(S)$ if and only if either $Y = X$ or $S \in \mathcal{W}$
Simple Self-Duality: E satisfies SIMP, REG, MAX

REMARK (i) Some of the properties listed above are not mutually independent: e.g. if $E(\varnothing) = \varnothing$ then (i) SUPA entails REG and N-MON, and (ii) CONV entails SUPA.

(ii) if E satisfies N-MON and SIMP it amounts in fact to what is usually denoted as a *Simple Game* (on X).

THE α-Effectivity Function and the β-Effectivity Function of a Strategic Game Form

DEFINITION (**α-Effectivity Function of a strategic game form)**
Let $G = (N, X, (S_i)_{i\in N}, h, (t_i(\succcurlyeq_i))_{i\in N})$ be a game in strategic form, and $[G] = (N, X, (S_i)_{i\in N}, h)$ its game form.
Then the *α-Effectivity Function* (α-EFF) of $[G]$ is the function $E^{\alpha}_{[G]} : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ defined as follows:
for every $T \subseteq N$,

$$E^{\alpha}_{[G]}(T) := \left\{ \begin{array}{l} Y \subseteq X : \text{there exists } s_T \in \prod_{i\in T} S_i \text{ such that} \\ h(s_T, t_{N\setminus T}) \in Y \text{ for every } t_{N\setminus T} \in \prod_{i\in N\setminus T} S_i \end{array} \right\}$$

REMARK The α-EFF $E^{\alpha}_{[G]}$ specifies the ability of a coalition $T \subseteq N$ to 'force' an event $Y \subseteq X$ as the capability of T to devise a coordinated strategy-profile for its members that engenders Y (namely, *some* outcome $x \in Y$) whatever the choice of strategies enacted by players *not* belonging to T.

DEFINITION (β-**Effectivity Function of a strategic game form)**
Let $G = (N, X, (S_i)_{i\in N}, h, (t_i(\succcurlyeq_i))_{i\in N})$ be a game in strategic form, and $[G] = (N, X, (S_i)_{i\in N}, h)$ its game form.
Then the β-*Effectivity Function* (β-EFF) of $[G]$ is the function $E^{\beta}_{[G]} : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ defined as follows:
for every $T \subseteq N$,

$$E^{\beta}_{[G]}(T) := \left\{ \begin{array}{c} Y \subseteq X : \text{for every } t_{N\setminus T} \in \prod_{i\in N\setminus T} S_i \text{ there exists } s_T \in \prod_{i\in N\setminus T} S_i \\ \text{such that } h(s_T, t_{N\setminus T}) \in Y \end{array} \right\}.$$

REMARK The β-EFF $E^{\beta}_{[G]}$ specifies the ability of a coalition $T \subseteq N$ to 'force' an event $Y \subseteq X$ as the capability of T to *counteract* to any possible choice of strategies by the other players by devising a *response* to the latter consisting of a coordinated strategy-profile for its members that engenders Y (namely, *some* outcome $x \in Y$).

BASIC PROPERTIES OF α-EFF and β-EFF AND TIGHTNESS OF STRATEGIC GAME FORMS

Claim Let $[G] = (N, X, (S_i)_{i \in N}, h)$ be a strategic game form. Then,
(i) $E^{\alpha}_{[G]} \leqslant E^{\beta}_{[G]}$,namely $E^{\alpha}_{[G]}(T) \subseteq E^{\beta}_{[G]}(T)$ for all $T \subseteq N$;
(ii) $E^{\alpha}_{[G]}$ satisfies MON, SV, REG, SUPA, but not necessarily MAX;
(iii) $E^{\beta}_{[G]}$ satisfies MON, SV, MAX but not necessarily REG or SUPA.

DEFINITION (**Tightness of strategic game forms)** A strategic game form $[G]$ is said to be **tight** if $E^{\alpha}_{[G]} = E^{\beta}_{[G]}$

EXAMPLE (Reprise)

Strategic Game Form of a 2×2 bimatrix game
$[G] = (N = \{1,2\}, X = \{a,b,c,d\}, (S_i = \{s_i, t_i\})_{i \in \{1,2\}}, h)$
with $h((s_1,s_2)) = a$, $h((s_1,t_2)) = b$,
$h((t_1,s_2)) = c$, $h((t_1,t_2)) = d$,

representable by the matrix- game form

1/2	t_1	t_2
s_1	a	b
s_2	c	d

Notice that $E^{\alpha}_{[G]}(\{1\}) = \uparrow \{\{a,b\},\{c,d\}\}$
, $E^{\alpha}_{[G]}(\{2\}) = \uparrow \{\{a,c\},\{b,d\}\}$, $E^{\alpha}_{[G]}(N) = \{Y \subseteq X : Y \neq \varnothing\}$.
(Notation: for any family $\mathcal{Y}$ of subsets of X, $\uparrow \mathcal{Y}$ denotes 'the collection of sets in $\mathcal{Y}$ and of all their supersets'),
and $E^{\beta}_{[G]} = E^{\alpha}_{[G]}$ i.e. $[G]$ is tight [Check it!].

EXAMPLE (Reprise)

Majority voting game form with pseudorandom endogenous president selection

Game form $\Gamma^{maj} = (N, X, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{maj})$,
as described above with
$h^{maj} : (X \times \mathbb{Z}_+)^N \rightarrow X$ defined by the following rules:
for each $s^N = ((x_1, z_1), ..., (x_n, z_n)) \in (L_X \times \mathbb{Z}_+)^N$,
$h^{maj}(s^N) = x$ if there exists precisely one $x \in X$ such that
$| \{ i \in N : x_i = x \} | \geq \frac{n}{2}$,
and $x_{i^*(s^N)}$ otherwise,
where $i^*(s^N) = \sum_{i \in N} z_i (\text{mod } n))$.

Claim $E^{\alpha}_{\Gamma^{maj}} = E^{\beta}_{\Gamma^{maj}}$ thus Γ^{maj} is tight, hence both $E^{\alpha}_{\Gamma^{maj}}$ and $E^{\beta}_{\Gamma^{maj}}$ satisfy MON, SV, REG, MAX, SUPA, AN, NT, SIMP. [Check it!]

EXAMPLE (Reprise)

Random Dictatorship voting game form

Game form $\Gamma^{rd} = (N, X, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{rd})$,
as described above

$N = \{1, ..., n\}$,

$X = \{x_1, ...x_m\}$

$h^{rd}(s^N) = x_{i^*(s^N)}$, with $i^*(s^N) = \sum_{i \in N} z_i (\mathrm{mod}\, n))$.

Claim (i) $E^{\alpha}_{\Gamma^{rd}}$ is such that $E^{\alpha}_{\Gamma^{rd}}(N) = \{Y \subseteq X : Y \neq \varnothing\}$, and $E^{\alpha}_{\Gamma^{rd}}(T) = \{X\}$ for any $\varnothing \neq T \subset N$;

(ii) $E^{\beta}_{\Gamma^{rd}}$ is such that $E^{\beta}_{\Gamma^{rd}}(T) = \{Y \subseteq X : Y \neq \varnothing\}$, for any $\varnothing \neq T \subseteq N$;

(iii) Thus, $E^{\alpha}_{\Gamma^{rd}}$ satisfies MON, SV, REG, SUPA, AN, NT, SIMP but not MAX, and $E^{\beta}_{\Gamma^{rd}}$ satisfies MON, SV, AN, NT, SIMP but not REG and SUPA. [Check it!]

Type-Revealing Strategic Game Forms and Strategy-Proofness

DEFINITION (**Type-Revealing Game Forms)** Let $[G] = (N, X, (S_i)_{i\in N}, h)$ be a strategic game form, T_i the space of possible types $t_i(\succcurlyeq_i)$ of player $i \in N$. Then, $[G]$ is *type-revealing for* $(T_i)_{i\in N}$ if for each $i \in N$ a *non-empty valued non-constant* correspondence $\tau_i : T_i \to\to S_i$, the *truthful type-revealing correspondence* of player i, is well-defined.

DEFINITION (**Strategy-Proof Game Forms)** Let $[G] = (N, X, (S_i)_{i\in N}, h)$ be a strategic game form which is type-revealing for $(T_i)_{i\in N}$, with $\tau = (\tau_i)_{i\in N}$ as its truthful type-revealing correspondence profile. Then, $[G]$ is **strategy-proof** on $\prod_{i\in N} T_i$ if for each $j \in N$, $(t_i(\succcurlyeq_i))_{i\in N} \in \prod_{i\in N} T_i$ and $s_j \in S_j$,

$$h((\tau_i(t_i(\succcurlyeq_i))_{i\in N}) \succcurlyeq_j h((s_j, (\tau_i(t_i(\succcurlyeq_i))_{i\in N\setminus\{j\}}))$$

i.e. 'telling the truth' is a *dominant strategy equilibrium*.

DEFINITION (**Coalitionally Strategy-Proof Game Forms**) Let $[G] = (N, X, (S_i)_{i \in N}, h)$ be a strategic game form which is type-revealing for $(T_i)_{i \in N}$, with $\tau = (\tau_i)_{i \in N}$ as its truthful type-revealing correspondence profile. Then, $[G]$ is **coalitionally strategy-proof** on $\prod_{i \in N} T_i$ if for each coalition $C \subseteq N$, $(t_i(\succcurlyeq_i))_{i \in N} \in \prod_{i \in N} T_i$ and $s_C \in \prod_{i \in C} S_i$, there exists $j \in C$ such that
$h((\tau_i(t_i(\succcurlyeq_i))_{i \in N}) \succcurlyeq_j h((s_{C \setminus \{j\}}, (\tau_i(t_i(\succcurlyeq_i))_{i \in (N \setminus C) \cup \{j\}}))$
i.e. 'telling the truth' is a *strong equilibrium.*

DEFINITION (**Aggregation Rule**) An **aggregation rule (AR)** on outcome space X is a strategic game form $[G] = (N, X, (S_i)_{i \in N}, h)$ with $S_i := X$ for each $i \in N$. In particular an AR is said to be:
(1) *Anonymous* if $h(s) = h(\sigma s)$ for every $s \in \prod_{i \in N} S_i$ and permutation $\sigma : N \rightarrow N$ (where $\sigma s = (s_{\sigma(1)}, ..., s_{\sigma(n)})$)
(2) *Idempotent* if for any $x \in X$, if $s = (x, ..., x)$ then $h(s) = x$.

DEFINITION (**Social Welfare Function) A social welfare function (SWF)** on outcome space X is an aggregation rule $[G] = (N, \mathcal{R}_X, (S_i)_{i \in N}, h)$ with $S_i := \mathcal{R}_X$ for each $i \in N$, where $\mathcal{R}_X$:=set of all total preorders on X.

DEFINITION (**Social Choice Function)** A **social choice function (SCF)** on outcome space X is a strategic game form $[G] = (N, X, (S_i)_{i \in N}, h)$ with $S_i := \mathcal{R}_X$ for each $i \in N$, where $\mathcal{R}_X$:=set of all total preorders on X.

REMARK Notice that every AR, SWF and SCF can be regarded as *type-revealing game forms* by a suitable formulation of the relevant type spaces and truthful type-revealing correspondences. [Check and provide the relevant details!].

DEFINITION (**Dictatorial Type-Revealing Game Forms)** A type-revealing game form $[G] = (N, X, (S_i)_{i \in N}, h)$ with truthful type-revealing correspondence profile $\tau = (\tau_i)_{i \in N}$ is **dictatorial** iff its strategic outcome function $h : \prod_{i \in N} S_i \longrightarrow X$ is *not* constant and there exists a player $i^* \in N$ such that for every $s = (s_i)_{i \in N}$, $h(s) \in top(\succcurlyeq_{i^*})$ where $t_i = t_i(\succcurlyeq_{i^*}) \in \tau_i^{-1}(s_i)$.

DEFINITION (**Constant Game Forms)** A game form $[G] = (N, X, (S_i)_{i \in N}, h)$ is **constant (**or trivial) iff its strategic outcome function $h : \prod_{i \in N} S_i \longrightarrow X$ is *constant.*

REMARK Dictatorial and constant ARs, SWFs, SCFs can be defined in a straightforward way, and turn out to be both *strategy-proof* and *coalitionally strategy-proof* under any natural specification of the players' type spaces.

THE TWO MAJOR LIMITATIVE THEOREMS OF SOCIAL CHOICE THEORY

Arrow's Impossibility Theorem for Social Welfare Functions: Let $[G] = (N, \mathcal{R}_X, (S_i)_{i \in N}, h)$ be a SWF such that $|X| \geq 3$, and $h : (\mathcal{R}_X)^N \to \mathcal{R}_X$ satisfies the following properties:

(i) *Independence of Irrelevant Alternatives (IIA):* for every $x, y \in X$ and every pair of preference profiles $(\succcurlyeq_i)_{i \in N}, (\succcurlyeq'_i)_{i \in N} \in (\mathcal{R}_X)^N$ such that $[x \succcurlyeq_i y$ if and only if $x \succcurlyeq'_i y$ for all $i \in N]$,

if $x \succcurlyeq y$ then $x \succcurlyeq' y$ (where $\succcurlyeq := h((\succcurlyeq_i)_{i \in N})$ and $\succcurlyeq' := h((\succcurlyeq'_i)_{i \in N})$)

(ii) *Weak Pareto Principle (WP):* for every $x, y \in X$ and every pair of preference profiles $(\succcurlyeq_i)_{i \in N}, (\succcurlyeq'_i)_{i \in N} \in (\mathcal{R}_X)^N$,

if $x \succ_i y$ for every $i \in N$ then $x \succ y$

(where $\succ$ denotes the asymmetric component of $\succcurlyeq := h((\succcurlyeq_i)_{i \in N})$).

Then h is *dictatorial* hence $[G]$ itself is a *dictatorial SWF.*

REMARK Clearly, a dictatorial SWF satisfies both IIA and WP: thus, this theorem amounts to *a characterization of dictatorial SWFs*. Notice that IIA can be regarded as a property which *rules out manipulation of outcomes through manipulation of the set of outcomes itself (*namely a requirement of *immunity to a sort of structural* -as opposed to *strategic-manipulation*).

Gibbard-Satterthwaite Impossibility Theorem for Social Choice Functions.
Let $[G] = (N, X, (S_i := \mathcal{R}_X)_{i \in N}, h)$ be a SCF with $|h[(\mathcal{R}_X)^N]| \geq 3$, and for any $i \in N$ take $T_i := \mathcal{R}_X$ defining the truthful type-revealing correspondence τ_i in the obvious way, namely $\tau_i(\succcurlyeq_i) := \succcurlyeq_i$for each $\succcurlyeq_i \in \mathcal{R}_X$. Then $[G]$ is *strategy-proof* if and only if it is a *dictatorial SCF.*

REMARK This theorem amounts to a characterization of dictatorial SCFs via *immunity to strategic manipulation.*

CLAIM The Random Dictatorship AR voting game form $\Gamma^{rd*} = (N, X \times \mathbb{Z}_+, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{rd})$ regarded as a type-revealing game form (with $T_i := \mathcal{R}_X$ $\tau_i(\succcurlyeq_i) := \succcurlyeq_i$for each $\succcurlyeq_i \in \mathcal{R}_X$, for any $i \in N$) is an anonymous, idempotent, strategy-proof and coalitionally strategy-proof AR. [Check it!]

MAJORITY RULE AND SINGLE PEAKED DOMAINS

DEFINITION (**Interval Space**) An *interval space* is a pair $\mathcal{I} = (X, I)$ where X is a set (e.g. the outcome space) and $I : X^2 \rightarrow \mathcal{P}(X)$ which satisfies the following conditions:
(*Symmetry*) for every $x, y \in X$, $I(x, y) = I(y, x)$;
(*Closedness*) for every $x, y \in X$, $I(x, y) \supseteq \{x, y\}$.

DEFINITION (**Median Interval Space**) An interval space $\mathcal{I} = (X, I)$ is *median* if for all $x, y, z \in X$

$$|I(x, y) \cap I(y, z) \cap I(x, z)| = 1.$$

DEFINITION (**Full Single Peaked Domain on a Median Interval Space**) Let $\mathcal{I} = (X, I)$ be a *median interval space.* Then, the *Full Single Peaked Domain* of total preorders on $\mathcal{I}$ is the set $D_{\mathcal{I}}$ of all total preorders on X *with a unique maximum* which satisfy the following two conditions:

(SP 1): for each $x \in X$ there exists a total preorder $\succcurlyeq \in D_{\mathcal{I}}$ having x as its unique maximum, denoted $x = top(\succcurlyeq)$.

(SP 2): for each $x, y, z \in X$ and $\succcurlyeq \in D_{\mathcal{I}}$ such that $x = top(\succcurlyeq)$, if $z \in I(x, y)$ then $z \succcurlyeq y$.

CLAIM Let $\mathcal{I} = (X, I)$ be a *median interval space.*Then, the *majority voting game form with pseudorandom endogenous president selection* $\Gamma^{maj} = (N, X, (S_i = X \times \mathbb{Z}_+)_{i \in N}, h^{maj})$ regarded as a type-revealing game form (with $T_i := D_{\mathcal{I}}$ and truthful type-revealing correspondence $\tau_i(\succcurlyeq_i) := \succcurlyeq_i$for each $\succcurlyeq_i \in D_{\mathcal{I}}$, for any $i \in N$) is an anonymous, idempotent and strategy-proof AR.

The Basic Two-Sided Matching Model as a Coalitional Game

The Basic Two-Sided Matching Model: Marriage-Game Interpretation

$$\mathcal{M} = (M, W, (\succcurlyeq_i)_{i \in M \cup W})$$

Two sets of agents: $M, W \neq \varnothing$, $M \cap W = \varnothing$

M set of men, W set of women

$\succcurlyeq_i$ preference relation of $i \in M \cup W$ with

$\succcurlyeq_i \subseteq (W \cup \{i\})^2$ if $i \in M$

$\succcurlyeq_i \subseteq (M \cup \{i\})^2$ if $i \in W$

For all $i \in M \cup W$, $\succcurlyeq_i$ is a linear order (i.e. an *antisymmetric* total preorder: $x \succcurlyeq_i y$ and $y \succcurlyeq_i x$ imply $x = y$).

A *matching* is a function $\mu : M \cup W \rightarrow M \cup W$ which satisfies:

(i) (two-sidedness) $\mu(i) \in W \cup \{i\}$ if $i \in M$,and $\mu(i) \in M \cup \{i\}$ if $i \in W$;

(ii) (involution) $\mu(\mu(i)) = i$ for all $i \in M \cup W$.

The set of all matchings of $\mathcal{M}$ is denoted by $\mathbb{M}$. A *blocking unit* for a matching μ is an agent $i \in M \cup W$ such that $i \succ_i \mu(i)$. A *blocking pair* for a matching μ is a pair $(m, w) \in M \times W$ such that both $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

A matching μ is *stable* if it has neither blocking units nor blocking pairs.

The Coalitional Two-Sided Matching Game

$G^{\mathcal{M}} = (M \cup W, \mathbb{M}, E^{\mathbb{M}}, (\widehat{\succcurlyeq}_i)_{i \in M \cup W})$
with $M, W, \mathbb{M}$ as defined in the two-sided matching model.
Next, define the following new terms: for any $S \subseteq M \cup W$ and $\mu \in \mathbb{M}$

$\mu_{|S} : S \to M \cup W$ such that $\mu_{|S}(i) := \mu(i)$ for every $i \in S$

$\mathbb{M}_S := \{\mu_S \in S^S : \mu_S \text{ is a matching of } (S, (\succcurlyeq_i)_{i \in S})\}$

$\mathbb{M}_{\mu_S} := \left\{\mu \in \mathbb{M} : \mu_{|S} = \mu_S \in \mathbb{M}_S\right\}$ for any $\mu_S \in \mathbb{M}_S$

Then, for all $S \subseteq M \cup W$

$$E^{\mathbb{M}}(S) := \left\{\uparrow \left\{\mu \in \mathbb{M} : \mu_{|S} = \mu_S \in \mathbb{M}_S\right\} : \mu_S \in \mathbb{M}_S\right\} = \\ = \{\uparrow \mathbb{M}_{\mu_S} : \mu_S \in \mathbb{M}_S\}.$$

$\widehat{\succcurlyeq}_i \subseteq \mathbb{M}^2$ is a total preorder s.t. for any $\mu, \mu' \in \mathbb{M}$,
$\mu \widehat{\succcurlyeq}_i \mu'$ iff $\mu(i) \succcurlyeq_i \mu'(i)$ (for every $i \in M \cup W$).

EXAMPLE

$(M = \{m_1, m_2, m_3\}, W = \{w_1, w_2, w_3\}, (\succcurlyeq_i)_{i \in M \cup W})$

where
$$\begin{aligned}
\succcurlyeq_{m_1} &:= w_2 \succ_{m_1} w_1 \succ_{m_1} w_3 \succ_{m_1} m_1 \\
\succcurlyeq_{m_2} &:= w_1 \succ_{m_2} w_3 \succ_{m_2} w_2 \succ_{m_2} m_2 \\
\succcurlyeq_{m_3} &:= w_1 \succ_{m_3} w_2 \succ_{m_3} w_3 \succ_{m_3} m_1 \\
\succcurlyeq_{w_1} &:= m_1 \succ_{w_1} m_3 \succ_{w_1} m_2 \succ_{w_1} w_1 \\
\succcurlyeq_{w_2} &:= m_3 \succ_{w_2} m_1 \succ_{w_2} m_2 \succ_{w_2} w_2 \\
\succcurlyeq_{w_3} &:= m_1 \succ_{w_3} m_3 \succ_{w_3} m_2 \succ_{w_3} w_3 \quad .
\end{aligned}$$

Consider the matchings μ, μ' whose graphs are

$$gr(\mu) = \left\{ \begin{array}{c} (m_1, w_1), (m_2, w_2), (m_3, w_3), \\ (w_1, m_1), (w_2, m_2), (w_3, m_3) \end{array} \right\} \text{ and}$$

$$gr(\mu') = \left\{ \begin{array}{c} (m_1, w_1), (m_2, w_3), (m_3, w_2), \\ (w_1, m_1), (w_2, m_3), (w_3, m_2) \end{array} \right\}, \text{ respectively.}$$

Then, μ is an *unstable* matching (having both(m_1, w_2) and (m_3, w_2) as *blocking pairs*), while μ' is a *stable* matching. [Check it!].

The Deferred Acceptance Protocol DA^M with M-proposals

Step 1: Each $m \in M$ proposes to his top alternative $top^1(\succcurlyeq_m)$ in $W \cup \{m\}$
Step 2: Each $w \in W$ chooses her best alternative in $\{m \in M : top^1(\succcurlyeq_m) = w\} \cup \{w\}$
Step 3: Each m whose proposal at Step 1 was rejected proposes to his second best alternative

$$top^2(\succcurlyeq_m) \text{ in } W \setminus \{top^1(\succcurlyeq_m)\} \cup \{m\}$$

Step 4: Each $w \in W$ chooses her best option in $\{m \in M : top^j(\succcurlyeq_m) = w, j = 1, 2\} \cup \{w\}$
and so on...
Stopping Rule: The Protocol stops when a Step is reached either without any rejection or with no new admissible proposals.

Theorem (Gale, Shapley (1962)) Let be $\mathcal{M} = (M, W, (\succsim_i)_{i \in M \cup W})$ a basic two-sided matching model, and $DA^M(\mathcal{M})$ the output of the DA^M protocol as applied to $\mathcal{M}$. Then $DA^M(\mathcal{M}) \in \overline{\mathbb{M}}(\mathcal{M})$, namely $DA^M(\mathcal{M})$ is a stable matching of $\mathcal{M}$.

PROOF. (By contraposition): Suppose not. Then, for some $\mathcal{M}$, $\mu = DA^M(\mathcal{M}) \in \mathbb{M}$ has either a blocking unit or a blocking pair. But by definition of $DA^M(\mathcal{M})$ a blocking unit i for μ implies a proposal of i which is inconsistent with her/his preferences in $\mathcal{M}$. Thus μ has a blocking pair $(m, w) \in M \times W$, namely both (i) $m \succ_w \mu(w)$ and (ii) $w \succ_m \mu(m)$ hold. Now, (i) entails $m \neq \mu(w)$ hence -by definition of DA^M- $w \succ_m \mu(m)$ implies that it must be the case that at some previous step m did make a proposal to w which was *rejected* by w in favor of an alternative option $x \in M \cup \{w\}$, $x \succ_w m$. Therefore, by the same argument, the final $DA^M(\mathcal{M})$-partner $\mu(w)$ of w must be such that $\mu(w) \succsim_w x \succ_w m$, whence by transitivity $\mu(w) \succ_w m$, a contradiction.

The Coalitional Game of an Abstract Economy

The *Coalitional Game* $G^{\mathcal{E}} = (N, X, E^{\mathcal{E}}, (\widehat{\succcurlyeq}_i)_{i \in N})$ *of an abstract economy* $\mathcal{E} = (N, (X_i := \mathbb{R}^k_+, \omega_i, \succcurlyeq_i \subseteq X_i \times X_i)_{i \in N})$ is defined as follows:

$X := \prod_{i \in N} X_i$, and for any $T \subseteq N$

$$E^{\mathcal{E}}(T) = \left\{ \uparrow Y : Y := (\{x_T\} \uparrow) \times \prod_{i \in N \setminus T} X_i \text{ with } x_T \in \prod_{i \in N \setminus T} X_i \text{ and } \sum_{i \in T} x_i \leq \sum_{i \in T} \omega_i \right\}$$

where $\uparrow Y := \{Z \subseteq X : Y \subseteq Z\}$

and, for any $i \in N$, $\widehat{\succcurlyeq}_i \subseteq X \times X$.

Solution rules for coalitional games

DEFINITION **Core of a game in coalitional form**

Let $G = (N, X, E, (t_i(\succcurlyeq_i))_{i \in N})$ be a game in coalitional form.
An outcome x is said to be G-dominated by $Y \subseteq X$ -written $Ydom_G x-$ if there exists $T \subseteq N$ such that both $Y \in E(T)$ and $y \succ_i x$ hold for each $i \in T$ and $y \in Y$.
Then the *core* of G is the set of outcomes defined as follows:

$Co(G) := \{x \in X : \text{there is no } Y \subseteq X \text{ such that } Ydom_G x\}$.

DEFINITION **VNM sets of a game in coalitional form**

Let $G = (N, X, E, (t_i(\succcurlyeq_i))_{i \in N})$ be a game in coalitional form.
A set $V \subseteq X$ is a *VNM stable set* of G if and only if the two following conditions hold (where dom_G is defined as for the core)
(i)(*Internal stability*) for any $x \in V$ there is no $Y \subseteq X$ such that $Ydom_G x$ and $V \cap Y \neq \varnothing$;
(ii)(*External stability*) for any $z \in X \setminus V$ there exists $Y \subseteq X$ such that both $Ydom_G z$ and $V \cap Y \neq \varnothing$.

EXAMPLE.

A simple majority distribution game

$G^{maj} = (N = \{1, 2, 3\}, X = \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}, E^{maj}, (\succcurlyeq_i)_{i \in N})$,

where -for any $T \subseteq N$-

$E^{maj}(T) = \{A \subseteq X : A \neq \varnothing$ and either $A = X$ or $|T| \geq 2\}$,

and for each $i \in N$, and $x, x' \in X$, $x \succcurlyeq_i x'$ if and only if $x_i \geq x'_i$.

Recalling that $x dom^{G_{maj}} y$ if and only if there exist $T \subseteq N$ and $A \subseteq X$ such that $x \in A \in E^{maj}(T)$ and $z \succ_i y$ for each $z \in A$ and $i \in T$, prove that (i) $Co(G^{maj}) = \varnothing$; (ii) $\{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$ is a VNM-stable set of G^{maj}.

PROPOSITION: For any basic two-sided matching model $\mathcal{M} = (M, W, (\succcurlyeq_i)_{i \in M \cup W})$, $\overline{\mathbb{M}}(\mathcal{M}) = Co(G^{\mathcal{M}})$.

PROOF. $\supseteq$. Suppose $\mu \notin \overline{\mathbb{M}}(\mathcal{M})$ i.e. there exists either a blocking unit $i \in M \cup W$ or a blocking pair $(m, w) \in M \times W$ for μ. Then, in both cases there exist a coalition $S \subseteq M \cup W$ (indeed, a trivial one in the first case) and another matching $\mu' \neq \mu$ in $\mathbb{M}$ such that $\mu'_{|S} \in \mathbb{M}_S$ and $\mu'(j) \succ_j \mu(j)$ for every $j \in S$. Therefore, by definition, $\mu' Dom^{G(\mathcal{M})} \mu$ hence $\mu \notin Co(G^{\mathcal{M}})$.
. $\subseteq$. Suppose now that $\mu \notin Co(G^{\mathcal{M}})$. Then, by definition, there exists at least one coalition $S \subseteq M \cup W$ and another matching μ' in $\mathbb{M}$ such that $\mu'_{|S} \in \mathbb{M}_S$ and $\mu'(j) \succ_j \mu(j)$ for every $j \in S$: take a set-inclusion minimal coalition having such a property. If $S = \{i\}$ for some $i \in M \cup W$ then i is a blocking unit of μ, by definition, whence μ is unstable namely $\mu \notin \overline{\mathbb{M}}(\mathcal{M})$. Otherwise, by construction, both $S \cap M \neq \varnothing$ and $S \cap W \neq \varnothing$, whence by minimality $S = \{m, w\}$ for some $m \in M$ and $w \in W$. It follows that $S = \{m, w\}$ is in fact a blocking pair of μ, and $\mu \notin \overline{\mathbb{M}}(\mathcal{M})$.

The core of the coalitional game of an abstract economy

Let $G^{\mathcal{E}} = (N, \prod_{i \in N} X_i, E^{\mathcal{E}}, (\widehat{\succcurlyeq}_i)_{i \in N})$ with

$\mathcal{E} = (N, (X_i := \mathbb{R}^k_+, \omega_i, \succcurlyeq_i \subseteq X_i \times X_i)_{i \in N})$

Then,

$$Co(G^{\mathcal{E}}) := \left\{ \begin{array}{c} x \in X : \text{for any } T \subseteq N \text{ and any } y_T \leq \sum_{i \in T} \omega_i \\ \text{there exist } i \in T \text{ and } y_{N|T} \in \prod_{i \in N \setminus T} X_i \text{ such that } x \widehat{\succcurlyeq}_i y \end{array} \right\}.$$

PROPOSITION Let $\mathcal{E} = (N, (X_i := \mathbb{R}^k_+, \omega_i, \succcurlyeq_i \subseteq X_i \times X_i)_{i \in N})$ be an abstract economy and $(p, (x_i)_{i \in N}) \in \mathbb{R}^k_+ \times \prod_{i \in N \setminus T} X_i$ a competitive equilibrium of $\mathcal{E}$ (namely $\sum_{i \in T} x_i \leq \sum_{i \in T} \omega_i$ and, for each $i \in N$, $x_i \succcurlyeq_i y_i$ for every $y_i \in B_{(p,\omega_i)} := \{z_i \in X_i : p \cdot z_i \leq p \cdot \omega_i\}$).
Moreover, let $G^{\mathcal{E}} = (N, \prod_{i \in N} X_i, E^{\mathcal{E}}, (\widehat{\succcurlyeq}_i)_{i \in N})$ with $(\widehat{\succcurlyeq}_i)_{i \in N}$ such that, for every $i \in N$, and $x, y \in \prod_{i \in N} X_i$, $x \,\widehat{\succcurlyeq}_i y$ if and only if $x_i \succcurlyeq_i y_i$. Then $x := (x_i)_{i \in N} \in Co(G^{\mathcal{E}})$.[Check it!]
Proof. (By contraposition: suppose not .....(exercise)

REMARK (A counterexample to the efficiency claim for general competitive equilibrium allocations)
Consider now the following 3-agent 2-good economy
$\mathcal{E} = (\{1,2,3\}, (X_i := \mathbb{R}^k_+, \omega_i, \succcurlyeq_i)_{i \in N})$ with agents' preferences represented by ordinal utility functions $u_{(1}(x_1) = \min\{x_{11}, x_{12}\}$,
$u_2(x_2) = \min\{\frac{1}{2}x_{21}, x_{22}\}$, $u_3(x_3) = x_{31} + 100x_{32}$,
$\omega_1 = (5,4), \omega_2 = (1,1), \omega_3 = (5,100)$,and $G^{\mathcal{E}}$ its coalitional game with preferences $(\widehat{\succcurlyeq}_i)_{i=1,2,3}$ where $\widehat{\succcurlyeq}_1$ and $\widehat{\succcurlyeq}_2$ are $\{1,2\}$-'*locally egalitarian*' (they both prefer to share equally their own aggregate resources) and $\widehat{\succcurlyeq}_3$ is on the contrary '*self-centered*'. Then, a competitive equilibrium allocation of $\mathcal{E}$ is *not efficient with respect to* $(\widehat{\succcurlyeq}_i)_{i=1,2,3}$, hence it does *not* belong to $Co(G^{\mathcal{E}})$.[Check it!].

REMARK

(i) whether any allocation x of $\mathcal{E}$ (including a general competitive equilibrium allocation) does or does not belong to $Co(G^{\mathcal{E}})$ depends on $(\widehat{\succcurlyeq}_i)_{i\in N}$ and its relationship to $(\succcurlyeq_i)_{i\in N}$;

(ii) whether any allocation x of $\mathcal{E}$ (including a general competitive equilibrium allocation) is *strictly efficient* (namely whether or not for every other allocation y of $\mathcal{E}$ it is the case that if $y\widehat{\succcurlyeq}_i x$ for every $i \in N$ then $x\widehat{\succcurlyeq}_i y$ for every $i \in N$) also depends on $(\widehat{\succcurlyeq}_i)_{i\in N}$ and its relationship to $(\succcurlyeq_i)_{i\in N}$.

(iii) It follows, in particular, that a general competitive equilibrium allocation is (strictly) efficient under *self-centered preferences* (namely, for every $i \in N$, and every $x, y \in X$, $x\widehat{\succcurlyeq}_i y$ if and only if $x_i \succcurlyeq_i y_i$), but *not otherwise!*

More generally, in order to establish whether preferences on allocations are invariably self-centered or otherwise, specific information on $(\widehat{\succcurlyeq}_i)_{i\in N}$ -as distinguished from $(\succcurlyeq_i)_{i\in N}$- is required. Such an information however is *not* provided by $\mathcal{E}$, hence *an augmentation of $\mathcal{E}$ to some new model $\widehat{\mathcal{E}} := (\mathcal{E}, (\widehat{\succcurlyeq}_i)_{i\in N})$ is needed to substantiate any statement concerning issues (i) (ii) and (iii) above.*

Other coalitional game formats: coalitional games in characteristic function form

Coalitional Games in Characteristic Function Form with Transferable Utility (TU coalitional games)

A TU coalitional game is a pair $\mathfrak{G}_{tu} = (N, v)$ where N -a nonempty, usually finite, set- denotes the set of players, and the *characteristic (or worth) function* v is a function $v : \mathcal{P}(N) \rightarrow \mathbb{R}$.

A TU coalitional game (actually, an entire class of TU coalitional games) $\mathfrak{G}^{E}_{TU} := (N, v^{E})$ can be attached to any game in coalitional form $G = (N, X, E, (t_i(\succcurlyeq_i))_{i \in N})$ such that $X = Y \times \mathbb{R}$ and $\succcurlyeq_i$is a *quasilinear total preorder* for every $i \in N$, hence representable by a real-valued utility function u_i (with $u_i(y, r_i) = v_i(y) + r_i$), by defining $v^{E} : \mathcal{P}(N) \rightarrow \mathbb{R}$ as follows: for every $T \subseteq N$,

$$v^{E}(T) := \max_{A \in E(T)} \min_{x \in A} \sum_{i \in T} u_i(x).$$

Coalitional Games in Characteristic Function Form with Nontransferable Utility (NTU coalitional games)

A NTU coalitional game is a pair $\mathfrak{G}_{ntu} = (N, V)$ where N -a nonempty, usually finite, set- denotes the set of players, and the *characteristic function* V is a function $V : \mathcal{P}(N) \to \bigcup_{T \subseteq N} \mathcal{P}(\mathbb{R}^T)$ such that, for every $T \subseteq N$, $V(T) \subseteq \mathbb{R}^T$ is both *closed* and *comprehensive* (i.e. such that for every $x, y \in \mathbb{R}^T$ if $x \in V(T)$ and $y \leqslant x$ then $y \in V(T)$), and possibly *upper bounded* as well.

A NTU coalitional game (actually, an entire class of NTU coalitional games) $\mathfrak{G}^E_{ntu} = (N, V^E)$ can be attached to any game in coalitional form $G = (N, X, E, (t_i(\succcurlyeq_i))_{i \in N})$ such that X is also a *comprehensive* (and possibly compact) subset of $\mathbb{R}^N$, and $\succcurlyeq_i$is a *continuous total preorder* for every $i \in N$ (hence representable by a continuous real-valued utility function u_i), by defining $V^E : \mathcal{P}(N) \to \bigcup_{T \subseteq N} \mathcal{P}(\mathbb{R}^T)$ as follows: for every $T \subseteq N$,

$$V^E(T) := \bigcup_{A \in E(T)} \left\{ z \in \mathbb{R}^T : z \leqslant (\min_{x \in A} u_i(x))_{i \in T} \right\}.$$

(I) Games in Extensive Form with Perfect Information and No Chance Moves

A **Game in Extensive Form with Perfect Information and No Chance Moves** is a structure

$$G = (N, X, P, \leqslant, p_0, a, f, (t_i(\succcurlyeq_i))_{i \in N}) \quad \text{where}$$

N set of *players*, $N \neq \varnothing$ and usually finite ($N = \{1, ..., n\}$)

X set/space of *outcomes*

P set of *positions*

$(P, \leqslant, p_0)$ *position poset*, a partially ordered set (i.e. $\leqslant$ is a reflexive, transitive and antisymmetric binary '*before than*' relation on positions), with a *minimum* p_0 denoting the *initial position*, a (possibly empty) subset $P_T := \{p \in P : p \text{ is } \leqslant\text{-maximal}\}$ of *terminal positions*, and a (possibly empty) set $\overline{P}^{\infty}$ of *maximal* admissible infinite chains of positions such that of course $P_T \cup \overline{P}^{\infty} \neq \varnothing$

$a : (P \setminus P_T) \to N$ the *move-assignment function*

$f : P_T \cup \overline{P}^{\infty} \to X$ the *positional outcome-function*

$t_i(\succcurlyeq_i)$ *type* of player i, $i \in N$, including his/her *preference relation* $\succcurlyeq_i \subseteq X \times X$ (usually taken to be a *total preorder* hence representable by an *utility function* u_i).

SOME USEFUL AUXILIARY NOTIONS

The **covering relation** $\ll$ of $(P, \leqslant)$: for any $p, p' \in P$, $p \ll p'$ iff $[p < p'$ and for any $q \in P$, $p < q \leqslant p'$ entails $q = p']$

(then p is said to be a *lower cover* of p', and p' is an *upper cover* of p)

The **upper cover** $\ll^p$of $p \in P$: $\ll^p := \{p' \in P : p \ll p'\}$

The **lower cover** $\ll_p$of $p \in P$: $\ll_p := \{p' \in P : p' \ll p\}$

The **principal order ideal** $\downarrow p$ *generated by* p: $\downarrow p := \{p' \in P : p' \leqslant p\}$

The **principal order filter** $\uparrow p$ *generated by* p: $\uparrow p := \{p' \in P : p \leqslant p'\}$

The **order filter** $\uparrow Q$ *generated by* $Q \subseteq P$:

$\uparrow Q := \{p \in P : q \leqslant p$ for some $q \in Q\}$

A **chain** *of* $(P, \leqslant)$ is a subset $P' \subseteq P$ such that for any $x, y \in P'$, either $x \leqslant y$ or $y \leqslant x$ holds.

(II) Games in Extensive Form with Perfect Information and Chance Moves

A **Game in Extensive Form with Perfect Information and Chance Moves** is a structure

$G = (N, L_X, P, \leqslant, p_0, P^C, (\pi_p)_{p \in P^C}, a, f_{P^C}, (t_i(\succcurlyeq_i))_{i \in N})$ where

N set of *players*, $N \neq \varnothing$ and usually finite ($N = \{1, ..., n\}$)

L_X set of lotteries on X set/space of *'pure' outcomes*

P set of *positions* and $(P, \leqslant, p_0)$ *position poset*, a partially ordered set with a *minimum* p_0 denoting the *initial position*, a (possibly empty) subset P_T of *terminal positions* and a (possibly empty) set $\overline{P}^{\infty}$ of maximal admissible infinite chains of positions

$P^C \subseteq (P \setminus P_T)$ set of *chance positions*

π_p a lottery on $\ll^p$ i.e. the upper cover of p, for each $p \in P^C$

$a : (P \setminus P_T) \to N \cup \{\pi_p : p \in P^C\}$ the *move-assignment function*, such that $a(p) = \pi_p$ if $p \in P^C$, and $a(p) \in N$ otherwise

$f_{P^C} : L_{P_T \cup \overline{P}^{\infty}} \to L_X$ the *positional expected outcome-function* where $L_{P_T \cup \overline{P}^{\infty}}$ denotes the set of lotteries induced by $(\pi_p)_{p \in P^C}$ on $P_T \cup \overline{P}^{\infty}$, and $t_i(\succcurlyeq_i)$ *type* of player i, $i \in N$, with his/her *preference relation* $\succcurlyeq_i \subseteq X \times X$ (e.g. a *total preorder* represented by EU function u_i).

REMARK The pair $(P^C, (\pi_p)_{p \in P^C})$ is also denoted as a **stochastic channel.** Thus, a game in extensive form with perfect information and chance moves amounts to supplementing a game in extensive form with perfect information and no chance moves with a suitable stochastic channel. It also follows that a game in extensive form with perfect information and no chance moves can be regarded as a *special case* of a game in extensive form with perfect information and chance moves which obtains precisely when $P^C = \varnothing$.

SOME FURTHER EXTENSIONS

A game in extensive form with perfect information and chance moves can be further specialized or further extended in several interesting ways.

SIMULTANEOUS MOVES OF SEVERAL PLAYERS: take a set $\mathcal{N} \subseteq 2^N$ of coalitions and a profile $\mathcal{G} := (\mathcal{G}_T)_{T \in \mathcal{N}}$ of sets of strategic games and *redefine the move-assignment function* as $a_{\mathcal{G}} : (P \setminus P^T) \to N \cup \bigcup_{T \in \mathcal{N}} \mathcal{G}_T$ so that certain *coalitions* of players $T \in \mathcal{N}$ are allowed to move simultaneously to play a strategic game (e.g. *repeated strategic games* typically require that adjustment). A game in extensive form with perfect information that is endowed with such a move-assignment function is usually denoted as a **game in extensive form with almost perfect information**.

INDEFINITELY REPEATED INTERACTIONS: Consider $\overline{P}^{\infty}$ under the following interpretation: for each maximal infinite chain $((p_m))_{m=0}^{\infty}$ of $(P, \leqslant)$ any even position p_{2k}(starting of course with p_0) denotes the initial position of interaction k and any odd position p_{2k+1} denotes the terminal position of interaction k that is reached at the end of that interaction. In particular, for *indefinitely repeated games* one typically obtains $f : \overline{P}^{\infty} \rightarrow X$ since $P^T = \varnothing$).

DEFINITION (**Subgames of a game in extensive form with perfect information and no chance moves)**
Let $G = (N, X, P, \leqslant, p_0, a, f, (t_i(\succcurlyeq_i))_{i\in N})$ be a game in extensive form with perfect information and no chance moves, and $p \in P$. Then, the **subgame** of G induced by p is the game
$G_p = (N_p, X_p, P_p, \leqslant_p, p, a_p, f_p, (t_i(\succcurlyeq_{i|X_p}))_{i\in N_p})$
where

$$
\begin{aligned}
N_p &:= \{i \in N : a(p') = i \text{ for some } p' \in (\uparrow p)\} \\
X_p &:= \{x \in X : f(p') = x \text{ for some } p' \in (\uparrow p)\} \\
P_p &:= \uparrow p \\
\leqslant_p &:= \leqslant_{|(\uparrow p)} \\
a_p &:= a_{|(\uparrow p)} \\
f_p &:= f_{|P^T \cap ((\uparrow p)} \\
\succcurlyeq_{i|X_p} &:= \succcurlyeq_i \cap (X_p \times X_p)
\end{aligned}
$$

REMARK In words, a subgame G_p of the extensive game G is the extensive game that is obtained from G by *ignoring* all of the positions p' such that p does *not* occur *before* p'.

DEFINITIONS

A (*simple*) **digraph** is a pair $D = (X, E)$ where X is a set and $E \subseteq X^2 \setminus \Delta_X$ (with $\Delta_X := \{(x, x) : x \in X\}$);

A **path** of digraph $D = (X, E)$ is a digraph $\pi = (Y, E')$ with $Y \subseteq X$, $E' \subseteq E$,

$$E' = \{(x_1, x_2), (x_2, x_3), ..., (x_{k-2}, x_{k-1}), (x_{k-1}, x_k), ..\}.$$

A **cycle** of digraph $D = (X, E)$ is a *finite path* $\pi = (Y, E')$ of D with $E' = \{(x_1, x_2), ..., (x_{k-1}, x_k)\}$ such that $x_1 = x_k$, for some integer $k \geq 2$.

A (*directed*) **tree** is a digraph $D = (X, E)$ that is both

(i) *connected*: for any $x, y \in X, x \neq y$ there exists a path $\pi = (Y, E')$ such that

$$E' = \{(x_1, x_2), ..., (x_{k-1}, x_k)\} \text{ with } x = x_1 \text{ and } y = x_k, \text{ and}$$

(ii) *acyclic:* D has no cycles.

The **length** of a finite path $\pi = (Y, E')$ with $E' = \{(x_1, x_2), (x_2, x_3), ..., (x_{k-2}, x_{k-1}), (x_{k-1}, x_k)\}$ is $k - 1$.

The **length** of a digraph is the length of its *longest* path (it is possibly infinite). The **horizon** of an extensive game G with position poset $(P, \leqslant)$ is the *length of digraph* $D = (P, <)$.

REMARK Observe that the fragment $(P, \ll)$ of a game G in extensive form may be regarded as a (simple) digraph, but need *not* be a (directed) *tree*.

CLAIM: Let $(P, \leqslant)$ be a partially ordered set, and $<$ the asymmetric component of $\leqslant$. Then, $(P, <)$ is a (directed) *tree* if and only if the following condition holds:
(Tree Property) for each $p \in P$ the principal order ideal $\downarrow p$ generated by p is a *chain* of $(P, \leqslant)$. [Prove it!]

REMARK Thus, a game in extensive form G with position poset $(P, \leqslant)$ embodies a **game tree** iff $(P, \leqslant)$ satisfies the Tree Property.

EXAMPLE 1

Sequential Battle of the Sexes (Informal description)

As for Battle of the Sexes in strategic form (see previous slide 7) except for the fact that now player 1 is the first to choose which event to attend, and it is then up to player 2 to choose whether to adapt to player 1's choice imitating it, or to attend to her most preferred event.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 1 (cont.)

Sequential Battle of the Sexes

$G = (N = \{1, 2\}, X, P, \leqslant, p_0, a, f, (\succcurlyeq_1, \succcurlyeq_2))$

where $X := \{x, y, v, z\}$

$P := \{p_0, p_1, p_2, p_3, p_4, p_5, p_6\}$

$\leqslant := (p_0, P) \cup \{(p_1, p_3)(p_1, p_4)(p_2, p_5)(p_2, p_6)\} \cup \Delta_{P \setminus \{p_0\}}$

where $(p_0, P) := \{(p_0, p) : p \in P\}$

$a : \{p_0, p_1, p_2\} \to \{1, 2\}$ is defined as follows: $a(p_0) = 1$, $a(p_1) = a(p_2) = 2$

$f : \{p_3, p_4, p_5, p_6\} \to X$ is defined as follows: $f(p_3) = x, f(p_4) = y, f(p_5) = v, f(p_6) = z$

$\succcurlyeq_1 := (x, z, y, v)$, $\succcurlyeq_2 := (z, x, y, v)$ (with outcomes in decreasing preference ordering).

EXAMPLE 2

The Chain-Store Game (Informal description)

Player 1 is a prospective new entrant in a certain local market, and has to choose whether to start her own local business, or give up. Player 2 is the encumbent (currently dominant firm in that local market), and is to choose in turn whether to react to that possible new entry with a mutually costly price rebate (dumping) or to give up and adapt to sharing the market with player 1, hence to the prospect of a significantly reduced market share for itself.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 2 (cont.)

Chain-Store Game

$G = (N = \{1, 2\}, X, P, \leqslant, p_0, a, f, (\succcurlyeq_1, \succcurlyeq_2))$
where $X := \{x, y, v, z\}$
$P := \{p_0, p_1, p_2, p_3, p_4, p_5, p_6\}$
$\leqslant := (p_0, P) \cup \{(p_1, p_3)(p_1, p_4)(p_2, p_5)(p_2, p_6)\} \cup \Delta_{P \setminus \{p_0\}}$
where $(p_0, P) := \{(p_0, p) : p \in P\}$
$a : \{p_0, p_1, p_2\} \rightarrow \{1, 2\}$ is defined as follows: $a(p_0) = 1$, $a(p_1) = a(p_2) = 2$
$f : \{p_3, p_4, p_5, p_6\} \rightarrow X$ is defined as follows:
$f(p_3) = x, f(p_4) = y, f(p_5) = v, f(p_6) = z$
$\succcurlyeq_1 := (x, [v, z], y)$, $\succcurlyeq_2 := (v, x, [y, z])$ (with outcomes in decreasing preference ordering)

EXAMPLE 3

The Centipede Game (Informal description)
Player 1 can either cash in immediately the proceeds of a certain activity or start a sequence of joint investments with player 2, making the previous proceeds immediately available to the latter project, with the understanding of a subsequent similar, reciprocating behaviour on the part of player 2 to continue the joint investment sequence. The relevant time horizon of the process, however, is of a prefixed length k (with $k = 100$ in the original version of the game, whence its name), and both the players know (and know that all know) that circumstance. Moreover, the last investor is not going to be specifically rewarded for her possible last investment effort.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 3 (cont.)

Centipede Game (4-Truncated Version)
$G = (N = \{1,2\}, X, P, \leqslant, p_0, a, f, (\succcurlyeq_1, \succcurlyeq_2))$
where $X := \{x, y, v, w, z\}$
$P := \{p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$
$\leqslant := (p_0, P) \cup \{(p_2, p_i) : i = 2, 3, ..., 8\} \cup \{(p_4, p_i) : i = 4, 5, 6, 7, 8\} \cup \{(p_6, p_i) : i = 6, 7, 8\}$
where $(p_0, P) := \{(p_0, p) : p \in P\}$
$a : \{p_0, p_2, p_4, p_6\} \rightarrow \{1, 2\}$ is defined as follows:
$a(p_0) = a(p_4) = 1$, $a(p_2) = a(p_6) = 2$
$f : \{p_1, p_3, p_5, p_7, p_8\} \rightarrow X$ is defined as follows:
$f(p_1) = x, f(p_3) = y, f(p_5) = v, f(p_7) = w, f(p_8) = z$
$\succcurlyeq_1 := (z, v, w, z, y)$, $\succcurlyeq_2 := (w, z, y, v, x)$ (with outcomes in decreasing preference ordering)

EXAMPLE 4

Sequential Bargaining with Alternating Offers and Time Discounting (Informal presentation)

Two players are trying to reach an agreement to share a valuable, private and perfectly divisible resource alternating their proposals to be accepted or rejected by the recipient, under time pressure since the value of the relevant resource is subject to a positive rate of decay.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 4 (cont.)

Sequential Bargaining with Alternating Offers and Time Discounting (A game with infinite horizon)

$G = (N, X, P, \leqslant, p_0, a, f, (\succcurlyeq_1, \succcurlyeq_2))$ where

$N := \{1, 2\}$

$X := \left\{((x_1, x_2), t) \in \mathbb{R}^2_+ \times \mathbb{Z}_+ : x_1 + x_2 = 1\right\} \cup \{x^*\}$

$P := \bigcup_{t \in \mathbb{Z}_+} P_t$ with

$P_0 := \{p_0\}$,

$P_1 := \left\{p_{\mathbf{x},0} : \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+, x_1 + x_2 = 1\right\}$ and

for any $t \in \mathbb{Z}_+, t > 1$,

$P_t := P_{t,A} \cup P_{t,R}$ with

$P_{t,A} := \left\{p_{\mathbf{x},t-1,A} : \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+, x_1 + x_2 = 1\right\}$

$P_{t,R} := \left\{p_{\mathbf{x},t} : \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+, x_1 + x_2 = 1\right\}$

$P_T := \bigcup_{1 < t \in \mathbb{Z}_+} P_{t,A}$

$\leqslant := (p_0, P) \cup \left(\bigcup_{0 \leq t' < t} (P_{t'}, P_{t,A})\right) \cup \left(\bigcup_{0 \leq t' < t} (P_{t',R}, P_{t,R})\right) \cup \Delta_{P \setminus \{p_0\}}$

[continues]

[continues]

where

$(p_0, P) := \{(p_0, p) : p \in P\}, (P', P'') := \{(p', p'') : p' \in P', p'' \in P''\}$

$$\Delta_{P\setminus\{p_0\}} = \{(p, p) : p \in P \setminus \{p_0\}\}$$

$a : (P_0 \cup P_1 \cup \bigcup_{1<t} P_{t,A}) \to \{1, 2\}$ is defined as follows:

$a(p_0) = 1$, $a(p) = 2$ if $p \in P_1$ and

for all $t > 1$, $a(p) = 1$ if $p \in P_{t,A}$ with t even, and $a(p) = 2$ if $p \in P_{t,A}$ with t odd

$f : (\bigcup_{1<t\in\mathbb{Z}_+} P_{t,A}) \cup P^\infty \to X$ where P^∞ denotes the set of all maximal infinite chains of $(P, \leqslant)$ is defined

as follows: $f(p) = ((x_1, x_2), t)$, if $p \in \bigcup_{1<t\in\mathbb{Z}_+} P_{t,A}$ is such that $p = p_{\mathbf{x},t-1,A} \in P_t$ with $\mathbf{x} = (x_1, x_2)$, and

$$f(p) = x^* \text{ if } p \in P^\infty$$

$\succcurlyeq_i$, $i = 1, 2$ is a total preorder on X which is:

(i) *continuous* with respect to the product topology $\tau \times \tau'$ where τ is the Euclidean metric topology on $\mathbb{R}^2_+$ and τ' the discrete topology on $\mathbb{Z}_+$

[continues]

(ii) such that $((x_1, x_2), t) \succcurlyeq_i ((x_1, x_2), t+1))$ for any $((x_1, x_2), t) \in X$

(iii) *stationary* i.e. for any $((x_1, x_2), t), ((y_1, y_2), t), ((y_1, y_2), t+1) \in X$,

$((x_1, x_2), t) \succcurlyeq_i ((y_1, y_2), t)$ iff $((x_1, x_2), 0) \succcurlyeq_i ((y_1, y_2), 0)$ and

$((x_1, x_2), t) \succcurlyeq_i ((y_1, y_2), t+1)$ iff $((x_1, x_2), 0) \succcurlyeq_i ((y_1, y_2), 1)$

(iv) such that $x \succcurlyeq_i x^*$ for all $x \in X$

(hence $\succcurlyeq_i, i = 1, 2$ is representable by a continuous utility function u_i^δ defined as follows: for any $x = ((x_1, x_2), t) \in X$, $u_i^\delta(x) = \delta^t u_i(x_1, x_2)$ where $\delta \in (0, 1)$ and $u_i : \mathbb{R}_+^2 \to \mathbb{R}$ is a continuous function).

EXAMPLE 5

Voting by Veto game forms and games (Informal description of the 3-player 4-outcome case)

Three players agree to make a collective choice among four alternative proposals including the status quo by taking turns in vetoing one alternative, to the effect of jointly selecting the only one surviving the sequence of their three vetoes.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 5 (cont.)

Voting by Veto game form (for $n = 3$ voters, $m = 4$ alternatives)
$G^V = (N = \{1, 2, 3\}, X, P, \leqslant, p_0, a, f)$
where $X := \{x, y, v, z\}$
$P := \{p_i : i \in \{0, 1, ..., 40\}\}$
$\leqslant := (p_0, P) \cup \{(p_1, p_i) : i = 1, 5, 6, 7, 17, ..., 22\} \cup$
$\{(p_2, p_i) : i = 2, 8, 9, 10, 23, ..., 28\} \cup$
$\cup \{(p_3, p_i) : i = 3, 11, 12, 13, 29, ..., 34\} \cup$
$\{(p_4, p_i) : i = 4, 14, 15, 16, 35, ..., 40\} \cup$
$\cup \{(p_5, p_i) : i = 5, 17, 18\} \cup \{(p_6, p_i) : i = 6, 19, 20\} \cup$
$\{(p_7, p_i) : i = 7, 21, 22\} \cup$
$\cup \{(p_8, p_i) : i = 8, 23, 24\} \cup \{(p_9, p_i) : i = 9, 25, 26\} \cup$
$\{(p_{10}, p_i) : i = 10, 27, 28\} \cup$
$\cup \{(p_{11}, p_i) : i = 11, 29, 30\} \cup \{(p_{12}, p_i) : i = 12, 31, 32\} \cup$
$\{(p_{13}, p_i) : i = 13, 33, 34\} \cup$
$\cup \{(p_{14}, p_i) : i = 14, 35, 36\} \cup \{(p_{15}, p_i) : i = 15, 37, 38\} \cup$
$\{(p_{16}, p_i) : i = 16, 39, 40\}$
where $(p_0, P) := \{(p_0, p) : p \in P\}$ [continues]

[cont.]

$a : \{p_i : i = 0, 1, ..., 16\} \to \{1, 2, 3\}$ is defined as follows:

$$a(p_0) = 1,$$
$$a(p_i) = 2 \text{ for } i = 1, 2, 3, 4,$$
$$a(p_i) = 3 \text{ for } i = 5, 6, ..., 15, 16$$

$f : \{p_i : i = 17, 18, ..., 39, 40\} \to X$ is defined as follows:

$$f(p_i) = x, \text{ for } i = 26, 28, 32, 34, 38, 40,$$
$$f(p_i) = y, \text{ for } i = 20, 22, 30, 33, 36, 39$$
$$f(p_i) = v, \text{ for } i = 18, 21, 24, 27, 35, 37,$$

$f(p_i) = z$, for $i = 17, 19, 23, 25, 29, 31$.

A **voting by veto game** is induced by game form G^V at any preference profile $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$
[consider e.g. $\succcurlyeq_1 = (xyvz)$, $\succcurlyeq_2 = (yxzv)$, $\succcurlyeq_3 = (zyxv)$.Compute the outcome under 'sincere' voting!).

EXAMPLE 6

Principal-Agent Game with Hidden Action (Informal description)
Player 1, a risk-neutral Principal, chooses a contract to submit to player 2, the Agent. Any contract specifies the Agent's task and its expected output (a deliverable whose features are observable and verifiable), and compensations $\overline{t}$ and $\underline{t}$, each one contingent on the type -'high-quality' $\overline{x}$ or 'low-quality' $\underline{x}$, respectively- of the observed output of the Agent. The quality of the Agent's output is in turn positively correlated with her *costly effort* which *increases the probability* of a 'high-quality' output from π_0 to π_1 (with $\pi_1 > \pi_0$), and if the Agent provides such an effort the expected value of the Principal's revenue $S(x)$ is indeed larger than the cost of effort to the former. The Agent may either sign the contract proposed by the Principal and then choose whether to provide or withdraw his effort while executing that contract, or turn down the offer to take another job for the standard competitive salary w.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 6 (cont.)

Principal-Agent Game with Hidden Action (simple binary version)

$G = (N = \{I, II\}, L_X, P, \leqslant, p_0, P^C, (\pi_p)_{p \in P^C}, a, f, (\succcurlyeq_I, \succcurlyeq_{II}))$

where I denotes the Principal, II denotes the Agent

L_X set of lotteries on $X := \{\overline{x}, \underline{x},\} \times \mathbb{R}^2_+$ with $\{\overline{x}, \underline{x},\} \in \mathbb{R}^2_+$, $\overline{x} > \underline{x}$,

$P := \{p_0\} \cup P_{II} \cup P^C \cup P_T$ where

$P_{II} := \left\{(\underline{t}, \overline{t}) : (\underline{t}, \overline{t}) \in \mathbb{R}^2_+\right\}$,

$P^C := \left\{p^C_i : i \in \mathbb{R}^2_+ \times \{(s, \overline{e}), (s, \underline{e})\}\right\}$,

$\pi_{p^C_i} := \overline{\pi}\overline{x} \oplus (1 - \overline{\pi})\underline{x}$ if $i \in i \in \mathbb{R}^2_+ \times \{(s, \overline{e})\}$ and

$\underline{\pi}\overline{x} \oplus (1 - \underline{\pi})\underline{x}$ if $i \in i \in \mathbb{R}^2_+ \times \{(s, \underline{e})\}$ with $\overline{\pi} > \underline{\pi}$

$P_T := \left\{p_i : i \in \mathbb{R}^2_+ \times \{(s, \overline{e}), (s, \underline{e}), r\}\right\}$ [continues]

[continues]

$\leqslant := (p_0, P) \cup (P_{II}, P^C) \cup (P_{II}, P_T) \cup (P^C, P_T)$

where $(p_0, P) := \{(p_0, p) : p \in P\}$,

$(P_{II}, P^C) := \{(p, p') : p \in P_{II}, p' \in P^C\}$,

$(P_{II}, P_T) := \{(p, p') : p \in P_{II}, p' \in P_T\}$,

$(P^C, P_T) := \{(p, p') : p \in P^C, p' \in P_T\}$

$a : \{p_0\} \cup P_{II} \cup P^C \rightarrow \{I, II\}$ is defined as follows: $a(p_0) = I$, $a(p) = II$ if $p \in P_{II}$, $a(p) = \pi_p$ if $p \in P^C$

$f : L_{P_T}(C) \rightarrow L_X$ where $L_{P_T}(C)$ denotes the set of lotteries on P_T induced by stochastic channel $(P^C, (\pi_p)_{p \in P^C})$

and for every $\pi \in L_{P_T}$,

$(f(\pi))(\overline{x}) = \pi(\{p_i : i \in \mathbb{R}^2_+ \times \{(s, \overline{e})\}\})$.

$\succcurlyeq_I$ the preference induced by maximization of the expected value of $(x - t)$ where $x \in \{\overline{x}, \underline{x}\}$, $t \in \{\overline{t}, \underline{t}\}$

$\succcurlyeq_{II}$ the preference induced by maximization of the expected value of $\max\{t, w\}$ where $t \in \{\overline{t}, \underline{t}\}$ and w is a competitive reserve wage that is available to II in case she rejects the proposed contract.

EXAMPLE 7

Burning Money Game: a bargaining game in extensive form with *almost perfect information* (Informal presentation)

Player 1 and player 2 agree that a joint exploitation of an indivisible resource is preferable to any individual appropriation, but propose two distinct distributions of the final proceeds of their possible cooperation (both claiming $\frac{3}{5}$ as their own share). In order to settle the ensuing bargaining process Player 1 enriches the interaction by introducing a previous unilateral signalling stage: she grabs a banknote from her wallet, lights a match and ostensibly stops to ponder whether to burn her money or not before starting the actual bargaining stage.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 7 (cont.)

Burning Money Game

$G = (N = \{1,2\}, X, P, \leqslant, p_0, a_{\mathcal{G}}, f, (\succcurlyeq_1, \succcurlyeq_2))$

where $X := \{x, y, v, z, a, b, c, d\}$

$P := \{p_i : i = 0,1,2,3,4,5,6,7,8,9,10\}$

$\leqslant := (p_0, P) \cup \{(p_1, p_j), j = 1,3,4,5,6\} \cup$
$\{(p_2, p_j); j = 2,7,8,9,10\} \cup \Delta_{P \setminus \{p_0, p_1, p_2\}}$

where $(p_0, P) := \{(p_0, p) : p \in P\}$

$a_{\mathcal{G}} : \{p_0, p_1, p_2\} \to N \cup \{G_N^I, G_N^{II}\}$ where $\mathcal{G} = \{G_N^I, G_N^{II}\}$

with $G_N^I :=$

1/2	s_2	t_2
s_1	$\frac{5}{2}; 2$	$-\frac{1}{2}; 1$
t_1	$\frac{1}{2}; 0$	$\frac{3}{2}; 3$

,

$G_N^{II} :=$

1/2	s_2	t_2
s_1	3; 2	0; 1
t_1	1; 0	2; 3

is defined as follows:

$a_{\mathcal{G}}(p_0) = 1,\ a_{\mathcal{G}}(p_1) = G_N^I,\ a_{\mathcal{G}}(p_2) = G_N^{II}$

[cont.]

[cont.]

$f : \{p_j, j = 3, 4, ..., 9, 10\} \to X$ is defined as follows:

$f(p_3) = x, f(p_4) = y, f(p_5) = v, f(p_6) = z$

$f(p_7) = a, f(p_8) = b, f(p_9) = c, f(p_{10}) = d$

(where

$x = h_{G_N^I}(s, s), y = h_{G_N^I}(s, t), v = h_{G_N^I}(t, s), z = h_{G_N^I}(t, t)$

$a = h_{G_N^{II}}(s, s), b = h_{G_N^{II}}(s, t), c = h_{G_N^{II}}(t, s), d = h_{G_N^{II}}(t, t)$,

and $h_{G_N^I}, h_{G_N^{II}}$ denote the strategic outcome functions of G_N^I and G_N^{II}, respectively)

$\succcurlyeq_1 := (a, x, d, z, c, v, b, y)$,

$\succcurlyeq_2 := ([d, z], [a, x], [b, y], [c, v])$

(with outcomes in decreasing preference ordering and $[., .]$ denoting indifference).

EXAMPLE 8

Indefinitely Repeated Prisoner Dilemma with Discounting (Informal description)

Player 1 and player 2 play indefinitely the Prisoner Dilemma game sharing at each interaction the expectation that there will be a next one, with probability $\delta \in (0, 1)$.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide an alternative a two-dimensional representation of the game by a labelled graph (or at least, try!).

EXAMPLE 8 (cont.)

Indefinitely Repeated Prisoner Dilemma with Discounting

$G = (N = \{1, 2\}, X, P, \leqslant, p_0, a_{\mathcal{G}}, f, (\succcurlyeq_1^{\delta}, \succcurlyeq_2^{\delta}))$

where

$X := (x^t : x^t \in h_{G_N^{PD}}(s_1, s_2), h_{G_N^{PD}}(s_1, t_2), h_{G_N^{PD}}(t_1, s_2), h_{G_N^{PD}}(t_1, t_2))_{t=1}^{\infty}$

$P := \{p_{mj} : m = 0, 1, 2, ..., \infty, j \in \{(s_1, s_2), (s_1, t_2), (t_1, s_2), (t_1, t_2)\}\}$

$\leqslant := \{(p_i, p_j), i, j = 0, 1, ..., \infty, i \leq j\}$

where $(p_0, P) := \{(p_0, p) : p \in P\}$

$a_{\mathcal{G}} : P \to N \cup \{G_N^{PD}\}$ where $\mathcal{G} = \{G_N^{PD}\}$

with $G_N^{PD} :=$

1/2	s_2	t_2
s_1	2; 2	0; 3
t_1	3; 0	1; 1

,

is the function defined as follows: for any $p_t \in P$

$a_{\mathcal{G}}(p_t) = G^t := G_N^{PD}$ [cont.]

[cont.]

$f : \overline{P}^{\infty} \to X$ is defined as follows:

for any $(p_m)_{m=0}^{\infty} \in \overline{P}^{\infty}$

$$\overline{P}^{\infty} := ((p_m, p_{mj}) : j \in \{(s_1, s_2), (s_1, t_2), (t_1, s_2), (t_1, t_2)\})_{m=1}^{\infty},$$

$$f((p_m)_{m=0}^{\infty}) = ((h_{G^t}(j)) : m = 2t + 1)_{t=1}^{\infty}$$

and for every $(x^t)_{t=1}^{\infty}, (y^t)_{t=1}^{\infty} \in X$ and $i = 1, 2$,

$$(x^t)_{t=1}^{\infty} \succcurlyeq_i^{\delta} (y^t)_{t=1}^{\infty}$$

if and only if $\sum_{t=1}^{\infty} \delta^{t-1} u_i(x^t) \geq \sum_{t=1}^{\infty} \delta^{t-1} u_i(y^t)$

(where $\delta \in (0, 1)$ and $u_i(.)$ is an utility function representing the preference relation $\succcurlyeq_i$ of i in G_N^{PD}, $i = 1, 2$).

The Game in Strategic Form induced by a Game in Extensive Form with Perfect Information and Chance Moves

$$G = (N, L_X, P, \leqslant, p_0, P^C, (\pi_p)_{p \in P^C}, a, f, (t_i(\succcurlyeq_i))_{i \in N})$$

is defined as follows:

$$G^S = (N, L_X, (S_i^G)_{i \in N}, h^G, (t_i(\succcurlyeq_i))_{i \in N}) \text{ where}$$

$$S_i^G := \left\{ \begin{array}{c} s_i : a^{-1}(i) \rightarrow \bigcup_{p \in a^{-1}(i)} \{p' \in P : p \ll p'\} \text{ such that} \\ s_i(p) \in \{p' \in P : p \ll p'\} \text{ for each } p \in a^{-1}(i) \end{array} \right\}$$

for each $i \in N$

Observe that by construction each strategy profile $s = (s_1, ..., s_n) \in \prod_{i \in N} S_i^G$ determines a *unique* path $\pi(s)$ of position digraph $(P, <)$ joining p_0 to some $p \in P_T$:the edges of the path that include a *chance* position $p_c \in P^C$ and another position in the upper cover of p_c come with a probability attached to them. The probability p contributed by $\pi(s)$ is the *product* of the probabilities of its edges including a chance position.

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$h^G : \prod_{i \in N} S_i^G \rightarrow L_X$ is defined as follows: for any $s = \prod_{i \in N} S_i^G$, $h^G(s)$ assigns to each 'pure' outcome $x \in X$ the *sum* of the probabilities contributed by the paths that (i) have non-zero probability according to s and (ii) end up at some terminal position $p \in P_T$ with $f(p) = x$.

REMARK In a game in extensive form 'strategy' is a *derivative* notion not a primitive one as in a game in strategic form. Thus, the notion of strategy has to be explicitly defined by means of the primitive terms of the relevant extensive game. The definition presented above amounts to specifying a strategy as a *complete action plan of a player* concerning the way of playing the game, where 'complete' stands for 'under any possible circumstance consistent with the rules of the game'.

MIXED EXTENSIONS OF GAMES IN EXTENSIVE FORM WITH PERFECT INFORMATION

The mixed extension of a game is about expanding the set of admissible actions available to players by allowing for *lotteries on actions*. But in an extensive game players may have *several* opportunities to make choices among available actions, rather than *just one* as in strategic games. Thus, there are apparently at least *two* natural ways of defining the mixed extension of an extensive game with perfect information (whether or not endowed with chance moves), namely introducing, respectively:

(1) MIXED STRATEGIES: *lotteries on strategies (i.e. on complete action plans)* namely $\widetilde{s}_i \in \widetilde{S}_i := L_{S_i}$, extending accordingly the original outcome set X to L_X, and the preferences to total preorders $\widetilde{\succsim}_i$ on L_X that are representable by EU functions u_i (for every player $i \in N$)

(2) BEHAVIOURAL STRATEGIES: *sequences of independent lotteries on local actions* namely $\widetilde{m}_i^p \in \widetilde{M}_i^p := L_{\ll^p}$for each $p \in a^{-1}(i)$ (for every player $i \in N$).

Accordingly, for any extensive game with perfect information G, one may define **two distinct games in strategic form** which describe *two* distinct sorts of *mixed extension* of G:
(1) the *strategic-mixed extension* $G^{\widetilde{S}}$, an 'indirect' mixed extension consisting of a mixed extension of the strategic game G^S induced by G, namely $G^S = (N, L_X, (\widetilde{S}_i^G)_{i \in N}, h^G, (t_i(\widetilde{\succcurlyeq}_i))_{i \in N})$
(2) the *behavioural-mixed extension* $G^{\widetilde{M}}$ of G, a 'direct' mixed extension of G, namely $G^{\widetilde{M}} = (N, L_X, (\widetilde{M}_i(G))_{i \in N}, h^G, (t_i(\widetilde{\succcurlyeq}_i))_{i \in N})$
where $\widetilde{M}_i(G) = \prod_{p \in a^{-1}(i)} \widetilde{M}_i^p$.

REMARK A result due to Kuhn (1953) implies that *for any extensive game with perfect information the two mixed extensions obtained through mixed strategies and behavioural strategies, respectively, are in fact* **equivalent**, in the following sense. For each mixed strategy $\widetilde{s}_i$ of every player there exists a behavioural strategy $\widetilde{m}_i^p$ of that player that -for each profile s_{-i} of pure strategies of the other players- when combined with the latter generates a lottery on outcomes that replicates the lottery on outcomes generated in turn by $\widetilde{s}_i$ when combined with s_{-i} (and conversely, for each behavioural strategy $\widetilde{m}_i^p$ there exists a mixed strategy $\widetilde{s}_i$ that replicates the lotteries on outcomes generated by $\widetilde{m}_i^p$, in the same sense).

(I) Games in Extensive Form with Imperfect Information and No Chance Moves

A **Game in Extensive Form with Imperfect Information and No Chance Moves** is a structure

$$G = (N, X, P, \leqslant, p_0, a, f, (\Pi_i)_{i\in N}, (M_I)_{I\in \cup_{i\in N}\Pi_i}, (t_i(\succcurlyeq_i))_{i\in N})$$

where

N set of *players*, $N \neq \varnothing$ and usually finite ($N = \{1, ..., n\}$)

X set/space of *outcomes*

P set of *positions*

$(P, \leqslant, p_0)$ *position poset,* a partially ordered set (i.e. $\leqslant$ is a reflexive, transitive and antisymmetric binary '*before*' relation on positions), with a *minimum* p_0 denoting the *initial position,* and a non-empty subset $P_T := \{p \in P : p \text{ is } \leqslant\text{-maximal}\}$ of *terminal positions*

$a : (P \setminus P_T) \to N$ the *move-assignment function*

$f : L_{P_T} \to X$ the *extended positional outcome-function*

$\Pi_i := \{I_{ij}; j = 1, ..., j(i)\}$ a partition of $a^{-1}(i)$, the *information partition* of player $i \in N$, whose blocks -the *information sets* I_{ij} of i- are such that $|\ll^p| = |\ll^q| := \gamma_{ij}$ for any $p, q \in I_{ij}$ (for each player $i \in N$)

[continues]

[continues]

$M_I := M_{ij} = \left\{ M_{I1}, ..., M_{I\gamma_{ij})} \right\}$ *uniform partition* of moves of information set $I_l := I_{ij}$, whose equal-sized blocks -*the move-types* M_{lm}, $m = 1, ..., \gamma_{ij}$- have cardinality $|I_{ij}|$,

$t_i(\succcurlyeq_i)$ *type* of player i, $i \in N$, including his/her *preference relation* $\succcurlyeq_i \subseteq X \times X$ (usually taken to be a *total preorder* hence representable by an *utility function* u_i, possibly an EU function if a mixed extension of G is being considered).

(II) Games in Extensive Form with Imperfect Information and Chance Moves

A **Game in Extensive Form with Imperfect Information and Chance Moves** is a structure

$G = (N, L_X, P, \leqslant, p_0, P^C, (\pi_p)_{p \in P^C}, a, f, (\Pi_i)_{i \in N}, (M_l)_{l \in \cup_{i \in N} \Pi_i}, t_i(\succcurlyeq_i)_{i \in N})$ where

$(P^C, (\pi_p)_{p \in P^C})$ is a *stochastic channel* as previously defined with $P^C \subseteq P \setminus P_T$

$a : (P \setminus P_T) \longrightarrow N \cup \{\pi_p : p \in P^C\}$ is the *move-assignment function*, such that $a(p) = \pi_p$ if $p \in P^C$, and $a(p) \in N$ otherwise

$t_i(\succcurlyeq_i)$ *type* of player i, $i \in N$, including his/her *preference relation* $\succcurlyeq_i \subseteq X \times X$ (a *total preorder* representable by an *expected utility function* u_i),

and the rest of its primitive terms are specified as for games in extensive form with imperfect information and no chance moves.

REMARK Formally, a game in extensive form with imperfect information and no chance moves can also be regarded as a special case of an extensive game with imperfect information and chance moves with $P^C = \varnothing$.

Imperfect vs Perfect Recall

It can be easily checked that in an extensive game with imperfect information a player may well know all the details of the game yet have *imperfect recall*, being *unable to recall some of her previous moves when playing* that game. If, on the contrary, it is assumed that each player can rely on *perfect recall* of her previous moves in any play of the game then such a property must be explicitly formulated and required.

DEFINITION **Games in Extensive Form with Imperfect Information and Perfect Recall**
A game G in extensive form with imperfect information (with or without chance moves) has *perfect recall* if for each player i of G and information set $I_{ij} \in \Pi_i$, and for any pair of positions p, $p' \in I_{ij}$, $\downarrow p \cap \Pi_i = \downarrow p' \cap \Pi_i$.

REMARK In words, Perfect Recall requires that each player i be always able to distinguish between two move-positions assigned to her, if such move-positions follow two different sequences of moves by i herself.

EXAMPLE 9

Selten's Horse Game (Informal description)
Player 1 has the opportunity to start a joint activity with players 2 and 3 either by inviting player 3 to perform a certain task or by inviting player 2 to choose whether to perform that task herself or to ask in turn player 3 to perform it. Player 3 can perform the task either efficiently or not, and knows that player 2 will ensure a better supervision than player 1 but -contrary to player 1- will never compensate her (costly) efficient performance. However, player 3 is unable to identify the player who is inviting her to perform the task.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 10 (cont.)

Selten's Horse Game

$G = (\{1,2,3\}, L_X, P, \leqslant, p_0, a, f, (\Pi_i)_{i\in N}, (M_I)_{I\in\cup_{i\in N}\Pi_i}, (\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3))$

where $X := \{x, y, v, w, z\}$

$P := \{p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$

$\leqslant := (p_0, P) \cup \{(p_1, p_i) : i = 1,3,4\} \cup \{(p_2, p_i) : i = 2,5,6,7,8\} \cup \{(p_5, p_i) : i = 5,6,7\}$

where $(p_0, P) := \{(p_0, p) : p \in P\}$

$a : \{p_0, p_1, p_2, p_6\} \to \{1,2,3\}$ is defined as follows: $a(p_0) = 1$, $a(p_1) = a(p_5) = 3$, $a(p_2) = 2$

$f : \{p_3, p_4, p_6, p_7, p_8\} \to X$ is defined as follows: $f(p_3) = x, f(p_4) = y, f(p_6) = v, f(p_7) = w, f(p_8) = z$

$\Pi_1 = \{\{p_0\}\}$, $\Pi_2 = \{\{p_2\}\}$, $\Pi_3 = \{\{p_1, p_5\}\}$

$M_{11} = \{\{p_0p_1\}, \{p_0p_2\}\}, M_{21} = \{\{p_2p_5\}, \{p_2p_6\}\}$,

$M_{31} = \{\{p_1p_3, p_5p_7\}, \{p_1p_4, p_5p_8\}\}$

[continues]

[continues]

$\succcurlyeq_1$, $\succcurlyeq_2$, $\succcurlyeq_3$ representable by expected utility functions u_1, u_2, u_3(respectively) such that

$u_1(x) = u_2(x) = u_3(x) = 4$,
$u_1(y) = u_2(y) = u_3(y) = 1, u_1(v) = u_2(v) = 3$,
$u_1(w) = u_2(w) = 5, u_1(z) = u_2(z) = u_3(z) = 2$,
$u_3(v) = u_3(w) = 4$.

EXAMPLE 10

Principal-Agent Game with Hidden Type-Information (Informal presentation)

Player 1, a risk-neutral Principal, knows that two types of agents -'higly capable' and 'fairly capable'- are available, and also knows their respective probabilities π and $1-\pi$. The unit cost $\overline{\theta}$ of the output for the 'highly capable' type is so low that her performance is just unachievable for the 'fairly capable' type. The Principal chooses a contract to submit to an Agent: the contract specifies the Agent's task and two observable standard output-types -'high quality' $\overline{x}$ and 'low quality' $\underline{x}$- of the Agent's actual performance, and their respective compensations $\overline{t}$ and $\underline{t}$. The Agent can either sign the contract and -if of the 'highly capable' type- choose whether to deploy her best productive talent in performing the required task or replicate the performance of the 'fairly capable' type, or just turn down the offer.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 10 (cont.)

Principal-Agent Game with Hidden Type-Information (simple binary version)

$G = (\{I, II\}, L_X, P, \leqslant, p_0, P^C, \pi_{p_0}, a, f, (\Pi_i)_{i=I,II}, (M_I)_{I \in \Pi_I \cup \Pi_{II}}, (\succcurlyeq_I, \succcurlyeq_{II}))$

where I denotes the Principal, II denotes the Agent

L_X set of lotteries on $X := \{\overline{x}, \underline{x}, \}$ with $\{\overline{x}, \underline{x}, \} \in \mathbb{R}^2_+$, $\overline{x} > \underline{x}$,

$P := \{p_0\} \cup P_I \cup P_{II} \cup P_T$ where

$P^C := \{p_0\}$

$P_I := \left\{p^{\overline{\theta}}, p^{\underline{\theta}}\right\}$

$P_{II} := \{p_c : c := ((\underline{t}, \underline{x}), (\overline{t}, \overline{x})) \in \mathbb{R}^4_+\}$,

$P_T :=$
$\left\{p^s_j : j \in \{(\underline{t}, \underline{x}) : (\underline{t}, \underline{x}) \in \mathbb{R}^2_+\} \cup \left\{p^s_j : j \in \{(\overline{t}, \overline{x}) : (\overline{t}, \overline{x}) \in \mathbb{R}^2_+\}\right\}\right\} \cup \{p^r(p_c) : p_c \in P_{II}\}$

[continues]

[continues]

$\leqslant := (p_0, P) \cup (P_{II}, P^C) \cup (P_{II}, P_T) \cup (P^C, P_T)$

where $(p_0, P) := \{(p_0, p) : p \in P\}$,

$(P_I, P_{II}) := \{(p, p') : p \in P_I, p' \in P_{II}\}$,

$(P_I, P_T) := \{(p, p') : p \in P_I, p' \in P_T\}$,

$(P_{II}, P_T) := \{(p, p') : p \in P_{II}, p' \in P_T\}$

$\pi_{p_0} := \overline{\pi}\overline{\theta} \oplus (1 - \overline{\pi})\underline{\theta}$ with $\overline{\theta} < \underline{\theta}$ ($\theta \in \{\overline{\theta}, \underline{\theta}\} \subseteq \mathbb{R}_+$ being a cost-efficiency parameter)

$a : \{p_0\} \cup P_{II} \cup P^C \to \{I, II\}$ is defined as follows: $a(p_0) = \pi_{p_0}$, $a(p) = I$ if $p \in P_I$, $a(p) = II$ if $p \in P_{II}$

$f : L_{P_T}(p_0) \to L_X$ where $L_{P_T}(p_0)$ denotes the set of lotteries on P_T induced by stochastic channel $(\{p_0\}, \pi_{p_0})$

and for every $\pi \in L_{P_T}$,

$(f(\pi))(\overline{x}) = \pi(\left\{p_j^s : j = (\overline{t}, \overline{x}) \text{ for some } (\overline{t}, \overline{x}) \in \mathbb{R}^2_+\right\})$.

$\Pi_I = \left\{I_{I_1} := \left\{p^{\overline{\theta}}, p^{\underline{\theta}}\right\}\right\}$, $\Pi_{II} = \{I_{II_{p_c}} = \{p_c\} : p_c \in P_{II}, \}$,

$M_{I_{I_1}} = \left\{\left\{p^{\overline{\theta}}p, p^{\underline{\theta}}p\right\}, p \in P_{II}\right\}$, $M_{I_{II_{p_c}}} = \{\{p_c p_t\} : p_t \in P_T\}$ for any $p_c \in P_{II}$,

[continues]

$\succcurlyeq_I$ the preference induced by maximization of the expected value of $(S(x) - t)$ where $x \in \{\overline{x}, \underline{x}\}$, $t \in \{\overline{t}, \underline{t}\}$ and $S(.)$ is an isotonic real-valued function,

$\succcurlyeq_{II}$ the preference induced by maximization of the expected value of $(t - \theta x)$ where $t \in \{\overline{t}, \underline{t}\}$, $\theta \in \{\overline{\theta}, \underline{\theta}\}$ and $x \in \{\overline{x}, \underline{x}\}$.

EXAMPLE 11

Sequential Team Coordination with an absent-minded player

(Informal presentation)

Player 1 and player 2 meet at a certain location since player 2 has to collect a parcel player 1 is bringing him. Player 1, however, is absent-minded and prone to going wrong.

EXERCISE: First check the formal description of this game in the language of the extensive format as presented in the next slide, and then provide a two-dimensional representation of the game by a labelled graph.

EXAMPLE 11 (cont.)

Sequential Team Coordination with an absent-minded player (A game with imperfect recall:[Check it!])

$G = (\{1,2\}, L_X, P, \leqslant, p_0, a, f, (\Pi_i)_{i\in N}, (M_I)_{I\in \cup_{i\in N}\Pi_i}, (\succcurlyeq_1, \succcurlyeq_2))$

where L_X set of lotteries on $X := \{x, y, v, z\}$

$P := \{p_0, p_1, p_2, p_3, p_4, p_5, p_6\}$

$\leqslant := (p_0, P) \cup \{(p_2, p_i) : i = 2,3,4,5,6\} \cup \{(p_4, p_i) : i = 4,5,6\}$

where $(p_0, P) := \{(p_0, p) : p \in P\}$

$a : \{p_0, p_2, p_4\} \to \{1,2\}$ is defined as follows: $a(p_0) = a(p_2) = 1$, $a(p_4) = 2$

$f : \{p_1, p_3, p_5, p_6\} \to X$ is defined as follows:

$f(p_1) = x, f(p_3) = y, f(p_5) = v, f(p_6) = z$

$\Pi_1 = \{\{p_0, p_2\}\}$, $\Pi_2 = \{\{p_4\}\}$

$M_{11} = \{\{p_0p_1, p_2p_4\}, \{p_0p_2, p_2p_3\}\}, M_{21} = \{\{p_4p_5\}, \{p_4p_6\}\}$,

$\succcurlyeq_1 = \succcurlyeq_2$ total preorders on L_X that are representable by EU functions u_1, u_2 such that

$\succcurlyeq_{1|X} = \succcurlyeq_{2|X} = (v[xyz])$ (preferences in decreasing order, $[.]$ denoting a non trivial indifference class)

The Game in Strategic Form induced by a Game in Extensive Form with Imperfect Information and Chance Moves

$G = (N, L_X, P, \leqslant, p_0, P^C, (\pi_p)_{p \in P^C}, a, f, (\Pi_i)_{i \in N}, (M_I)_{I \in \cup_{i \in N} \Pi_i}, t_i(\succcurlyeq_i)_{i \in N})$

is defined as follows:

$$G^S = (N, L_X, (S_i^G)_{i \in N}, h^G, (t_i(\succcurlyeq_i))_{i \in N}) \text{ where}$$

$$S_i^G := \left\{ \begin{array}{c} s_i : \Pi_i \to \bigcup_{I_{ij} \in \Pi_i} M_{ij} \text{ such that} \\ s_i(I_{ij}) \in M_{ij} \text{ for each } I_{ij} \in M_{ij} \end{array} \right\} \text{ for each } i \in N$$

namely, *a strategy is a selection of a move-type for each information set* of the relevant player.

Observe that by construction -as for perfect information extensive games- each strategy-profile

$$s = (s_1, ..., s_n) \in \prod_{i \in N} S_i^G \text{ determines a } unique \text{ path } \pi(s) \text{ of}$$

position digraph $(P, <)$ joining p_0 to some $p \in P_T$:the edges of the path that include a *chance* position $p_c \in P^C$ and another position in the upper cover of p_c come with a probability attached to them. The probability p contributed by $\pi(s)$ is the *product* of the probabilities of its edges including a chance position.

[continues]
$h^G : \prod_{i \in N} S_i^G \rightarrow L_X$ is defined as follows: for any $s = \prod_{i \in N} S_i^G$, $h^G(s)$ assigns to each 'pure' outcome $x \in X$ the *sum* of the probabilities contributed by the paths that (i) have non-zero probability according to s and (ii) end up at some terminal position $p \in P_T$ with $f(p) = x$.

REMARK The definition presented above amounts to specifying a strategy as a *complete action plan of a player* concerning the way of playing the game, where 'complete' stands for 'under any possible circumstance consistent with the rules of the game'. The only difference with respect to perfect information extensive games is that *under imperfect information players can in general only select move-types rather than moves*.

MIXED EXTENSIONS OF GAMES IN EXTENSIVE FORM WITH IMPERFECT INFORMATION

As observed previously, in extensive games there are at least *two* natural ways of defining the mixed extension of an extensive game with perfect information (whether or not endowed with chance moves). The same can be done for *extensive games with imperfect information (whether or not endowed with chance moves)* by introducing, respectively:

(1) MIXED STRATEGIES: *lotteries on strategies (i.e. on complete action plans)* namely $\widetilde{s}_i \in L_{S_i^G}$, extending accordingly the original outcome set X to L_X, and the preferences to total preorders $\widetilde{\succcurlyeq}_i$ on L_X that are representable by EU functions u_i (for every player $i \in N$)

(2) BEHAVIOURAL STRATEGIES: *sequences of independent lotteries* on *local actions,* that is on *choices of move-types,* namely $\widetilde{m}_i = ((\widetilde{m}_{ij}) : I_{ij} \in \Pi_i)$ with $\widetilde{m}_{ij} \in \widetilde{M}_{ij} := L_{M_{ij}}$for each $I_{ij} \in \Pi_i$ (for every player $i \in N$).

Thus, we have *two distinct mixed extensions of a game in extensive form G with imperfect information*, defined as follows:

DEFINITION **Strategic-Mixed Extension of a Game G in Extensive Form with Imperfect Information**
The *strategic-mixed extension* $G^{\widetilde{S}}$, is the mixed extension of the strategic game G^S induced by G, namely the *game in strategic form*
$G^{\widetilde{S}} = (N, L_X, (\widetilde{S}_i^G)_{i \in N}, h^G, (t_i(\widetilde{\succsim}_i))_{i \in N})$

DEFINITION **Behavioural-Mixed Extension of a Game G in Extensive Form with Imperfect Information**
The *behavioural-mixed extension* $G^{\widetilde{M}}$ of G is the *game in strategic form*
$G^{\widetilde{M}} = (N, L_X, (\widetilde{M}_i(G))_{i \in N}, h^G, (t_i(\widetilde{\succsim}_i))_{i \in N})$
where $\widetilde{M}_i(G) = \prod_{I_{ij} \in \Pi_i} \widetilde{M}_{ij}$.

REMARK A classic theorem due to Kuhn (1953) establishes that *for any extensive game with imperfect information and* **perfect recall** *the two mixed extensions obtained through mixed strategies and behavioural strategies, respectively, are in fact* **equivalent**, in terms of outcomes (in the sense explained above for perfect information extensive games). Namely, for each *mixed strategy* $\widetilde{s}_i$ of every player there exists a *behavioural strategy* $\widetilde{m}_i$ of that player that -for each profile s_{-i} of pure strategies of the other players- when combined with the latter generates a lottery on outcomes that replicates the lottery on outcomes generated in turn by $\widetilde{s}_i$ when combined with s_{-i} (and conversely, for each behavioural strategy $\widetilde{m}_i$ there exists a mixed strategy $\widetilde{s}_i$ that replicates the lotteries on outcomes generated by $\widetilde{m}_i$, in the same sense). On the contrary, *in extensive games with imperfect information and imperfect recall such an outcome-equivalence between mixed and behavioural strategies* **may fail.** Thus, it may happen that at certain strategy profiles some behavioural strategy induces a lottery on outcomes which cannot be replicated by any mixed strategy (*this is the case for Sequential Team Coordination with an absent-minded player* presented above [**Check it!**]), or the converse holds, or both things happen to be true.

Solution Rules for Games in Extensive Form

Notice that given an extensive game G -with either perfect or imperfect information- it is always possible to compute its solutions through the following approach:
(i) *define its own game in strategic form* G^S and possibly its mixed extension $G^{\widetilde{S}}$, according to the rules introduced above, or
(ii) *define its behavioural-mixed extension* $G^{\widetilde{M}}$, which is another *game in strategic form* 'naturally' induced by G
and then
(iii) **use a solution rule for general strategic games** to solve G^S,$G^{\widetilde{S}}$ or $G^{\widetilde{M}}$.

REMARK Arguably, games in extensive form provide a promising tool for the analysis of *coalition formation and decay processes*. Yet, for several reasons that need not detain us here, *the study of cooperative solution rules for extensive form games is largely underdeveloped.* Accordingly, in the present notes we shall focus on **non-cooperative solution rules**.

Basic Non-cooperative Solution Rules for Games in Extensive Form

DEFINITION **Nash Equilibrium of a game in extensive form**

Let G be a game in extensive form. A *Nash equilibrium* of G is a Nash equilibrium of its game in strategic form G^S.

DEFINITION **Mixed (Nash) Equilibrium of a game in extensive form**

Let G be a game in extensive form. A *mixed (Nash) equilibrium* of G is a mixed (Nash) equilibrium of the strategic-mixed extension $G^{\widetilde{S}}$ of its game in strategic form G^S.

DEFINITION **Behavioural (Nash) Equilibrium of a game in extensive form**

Let G be a game in extensive form. A *behavioural (Nash) equilibrium* of G is a Nash equilibrium of the behavioural-mixed extension $G^{\widetilde{M}}$.

REMARK Notice, however, that the solution rules for general strategic games -by construction- are designed to work for any strategic game of a large class which is typically *not* restricted to strategic games arising from extensive games such as $G^S, G^{\widetilde{S}}, G^{\widetilde{M}}$. At the same time, $G^S, G^{\widetilde{S}}, G^{\widetilde{M}}$ *do carry some specific structure* coming from their 'extensive origin'. Thus, it makes sense to try and exploit precisely that structure to develop further sensible solution rules for them in order to predict the 'stable' outcomes of G.

Thus, we turn to a couple of basic non-cooperative solution rules that exploits the additional structure of an extensive game to refine Nash equilibrium and based on (two variants of) the notion of *subgame perfection of (Nash) equilibria.*

DEFINITION **Subgame Perfect (Nash) Equilibrium of a Game in Extensive Form**
Let $G = (N, L_X, P, \leqslant, p_0, P^C, (\pi_p)_{p \in P^C}, a, f, (t_i(\succcurlyeq_i))_{i \in N})$ be a game in extensive form with perfect information and chance moves. Then a **subgame perfect (Nash) equilibrium** of G is a strategy profile $s = (s_1, ..., s_n) \in \prod_{i \in N} S_i^G$ of the strategic game
$G^S = (N, L_X, (S_i^G)_{i \in N}, h^G, (t_i(\succcurlyeq_i))_{i \in N})$ induced by G such that for every $p \in P \setminus P_T$,
$s^p = (s_1^p, ..., s_n^p)$ is a Nash equilibrium of the strategic game G_p^S induced by subgame G_p of G (where s_i^p denotes the *continuation strategy of* s_i in G_p i.e. the restriction of s_i to G_p).

REMARK A subgame perfect Nash equilibrium may be described as a Nash equilibrium that does *not* rely on any *threat* or *promise* which is *not credible.*

COMPUTING SUBGAME PERFECT EQUILIBRIA BY BACKWARD INDUCTION: THE ZERMELO-KUHN ALGORITHM

DEFINITION **The Zermelo-Kuhn algorithm**
Let $G = (N, L_X, P, \leqslant, p_0, P^C, (\pi_p)_{p \in P^C}, a, f, (t_i(\succcurlyeq_i))_{i \in N})$ be a game in extensive form with perfect information and chance moves, having the **tree property** and with a **finite horizon.**
Step 0. Select a path π of maximum length of the position digraph $D = (P, <)$: clearly, by construction, the end-points of π are p_0 and a terminal position $\overline{p} \in P_T$.
Step 1. Move back from $\overline{p}$ to its lower cover p' (which is *unique*, because of the tree property), put $p' = p^*$ if $p' \notin P^C$ (if $p' \in P^C$ move back to the lower cover p'' of p' and so on until a $\widetilde{p} \notin P^C$ is reached: then put $\widetilde{p} = p^*$; if that does not happen before p_0 is reached, then move back to Step 0 and choose a *new path* π' of $(P, \leqslant)$ of *maximum length among those different from* π).
[continues]

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Step 2. Consider the upper cover $\ll^{p^*}$ of p^*, *choose* - among the paths joining p^* to a terminal position or a lottery on terminal positions through a position of $\ll^{p^*}$- *one of those with maximum expected utility value for player* $i^* := a^{-1}(p^*)$, and *store* that path in a special register.

Step 3. Build up the *reduced game* $G_{-p^*} := G_{-p^*}(G)$ obtained by replacing p^* with the terminal position -or lottery on terminal positions- of the path selected at Step 2, replace G with $G_{-p^*}(G)$, and restart from Step 0 as applied to $G_{-p^*}(G)$ with its position digraph. Stop when the position digraph of the current $G_{-p^*}$is reduced to $(\{p_0\}, \varnothing)$.

Comment By suitably 'gluing' together the adjacent stored paths of a complete run of the algorithm, at least one path -and typically *several mutually disconnected paths*- of the position digraph $(P, <)$ of G are singled out, including -by construction- a *unique path joining the initial position to a* (possibly degenerate) *lottery on terminal positions of* G. It can be proved that such *a* (possibly degenerate) lottery is a *subgame perfect equilibrium outcome* of G, and a *subgame perfect strategy-profile* which generates that outcome can also be immediately identified.

[continues]

[continues]
Re-running the algorithm for each one of the possibly several optimal choices available at some positions the set of *all subgame perfect equilibria is thus determined.*

REMARK The Zermelo-Kuhn algorithm amounts to a *multiagent version of dynamic programming*. Observe that in the above presentation of the Zermelo-Kuhn algorithm P is not required to be finite. However, if P is *not* finite then suitable requirements of *compactness* on the upper covers $\ll^P$ and on the outcome space X, and of *continuity* on positional outcome-function f and preferences $\succcurlyeq_i$ of G are typically to be made, in order to make sure that all the maximization problems involved do admit solutions. On the other hand, it can be shown that the Zermelo-Kuhn algorithm can be easily adapted to the case of extensive games with position digraphs which are *not* trees. Moreover, a similar backward induction approach to computation of subgame perfect equilibria also works for certain extensive games with *infinite horizon* provided that preferences are continuous (including Sequential Bargaining with Alternate Offers and Time Discounting as introduced previously in these notes).

REMARK Notice that the Zermelo-Kuhn algorithm can be successfully used to obtain the solutions of both the Principal-Agent games (including the Hidden Type-Information one!). Yet, that algorithm is typically *not* suitable for extensive games with *imperfect information* because it treats every non-terminal position of an extensive game G as the initial position of a subgame of G, namely G_p. However, if the extensive game G is a game with *imperfect information* then it may be the case that such a p belongs to a *non-trivial information set*, namely to an information set that also includes *other non-terminal positions* without any information on their respective probabilities. But then, it does *not* make sense to accept/treat G_p as a *proper/relevant* subgame of G since the player in p does not know what is the probability she is going to start her play of G_p, or of some *other subgame* of G. It follows that the proper/relevant subgames of an extensive game with imperfect information may be too few to enable the formulation of a well-defined solution rule. In order to circumvent that problem, we turn now to another basic noncooperative solution rule which is based on the behavioural-mixed extension of an extensive game G and on a *suitably adapted notion of subgame-perfection that also works when G is an extensive game with imperfect information*.

DEFINITION **Perfect Behavioural Equilibrium of a game in extensive form with perfect information**

Let G be a game in extensive form with perfect information, and $G^{\widetilde{M}} = (N, L_X, (\widetilde{M}_i(G))_{i\in N}, h^G, (t_i(\widetilde{\succcurlyeq}_i))_{i\in N})$ its behavioural-mixed extension. Then, a **perfect behavioural equilibrium** of G is a strategy-profile $\widetilde{m} = (\widetilde{m}_i)_{i\in N} \in \prod_{i\in N} \widetilde{M}_i(G)$ (with $\widetilde{M}_i(G) := \prod_{p\in a^{-1}(i)} \widetilde{M}_i^p$, hence $\widetilde{m}_i := (\widetilde{m}_i^p)_{p\in a^{-1}(i)}$) such that for every $p \in P \setminus (P^C \cup P_T)$ $\widetilde{m}| \uparrow p = ((\widetilde{m}_i| \uparrow p))_{i\in N}$ is a Nash equilibrium of the behavioural-mixed extension $G_p^{\widetilde{M}}$ of subgame G_p of G (where $(\widetilde{m}_i| \uparrow p)$ denotes the *continuation behavioural strategy of* $\widetilde{m}_i$ in G_p i.e. the restriction of $\widetilde{m}_i$ to G_p).

The extension of perfect behavioral equilibrium to extensive games with imperfect information requires the following preliminary definition.

DEFINITION (**Proper subgames of a game in extensive form with imperfect information)**
Let
$G = (N, L_X, P, \leqslant$
$, p_0, P^C, (\pi_p)_{p\in P^C}, a, \widetilde{f}, (\Pi_i)_{i\in N}, (M_I)_{I\in\cup_{i\in N}\Pi_i}, t_i(\widetilde{\succcurlyeq}_i)_{i\in N})$
be a game in extensive form with imperfect information and chance moves, and $I^* := I_{ij} \in \Pi_i$ for some $i \in N$ a **disentangled** information set i.e. $p \in (\uparrow I^*) \cap I$ for some information set $I \in \cup_{i\in N}\Pi_i$ only if $I \subseteq \uparrow I^*$.Then, the **proper subgame** of G induced by I^*at behavioural strategy profile $\widetilde{m} = (\widetilde{m}_i)_{i\in N}$ is the game
$G_{I^*(\widetilde{m})} = (N_{I^*(\widetilde{m})}, X_{I^*(\widetilde{m})}, P_{I^*(\widetilde{m})}, \leqslant_{I^*(\widetilde{m})}, p_{I^*(\widetilde{m})}, P^C \cup$
$\left\{p_{I^*(\widetilde{m})}\right\}, (\pi_p)_{p\in P^C} \cup \{\widetilde{m}_{I^*}\}, a_{I^*(\widetilde{m})}, \widetilde{f}_{I^*(\widetilde{m})}, (t_i(\widetilde{\succcurlyeq}_{i|I^*}))_{i\in N_{I^*(\widetilde{m})}})$
where

$$N_{I^*(\widetilde{m})} := \{j \in N : a(p') = j \text{ for some } p' \in (\uparrow I^*)\}$$
$$X_{I^*(\widetilde{m})} := \{x \in X : f(p') = x \text{ for some } p' \in (\uparrow I^*)\}$$
$$P_{I^*(\widetilde{m})} := ((\uparrow I^*) \setminus I^*) \cup \left\{p_{I^*(\widetilde{m})}\right\} \text{ where } p_{I^*(\widetilde{m})} := \widetilde{m}_{ij}$$

$$\leqslant_{I^*(\widetilde{m})} := \leqslant \cap \left\{ (p_{I^*(\widetilde{m})}, p) : p \in (\uparrow I^*) \setminus I^* \right) \cup \left\{ p_{I^*(\widetilde{m})} \right\} \right\}$$

$$a_{I^*(\widetilde{m})} : ((\uparrow I^*) \setminus I^*) \cup \left\{ p_{I^*(\widetilde{m})} \right\} \rightarrow N_{I^*(\widetilde{m})} \text{ such that}$$

$$a_{I^*(\widetilde{m})}(p) = a(p) \text{ for any } p \in ((\uparrow I^*) \setminus I^*), \text{ and}$$

$$a_{I^*(\widetilde{m})}(p_{I^*(\widetilde{m})}) = i$$

$$\widetilde{f}_{I^*(\widetilde{m})} := \widetilde{f}_{|P^T \cap (\uparrow I^*)}$$

$$\succcurlyeq_{i|_{I^*(\widetilde{m})}} := \succcurlyeq_i \cap (P^T \cap (\uparrow I^*))^2$$

REMARK In words, for any information set I_{ij} of G, the proper subgame G_{I^*} of the extensive game G is the extensive game that is obtained from G by *ignoring* all of the positions p' such that there is no $p \in I_{ij}$ such that p occurs *before* p'.

DEFINITION **Perfect Behavioural Equilibrium of a game in extensive form with imperfect information**

Let G be a game in extensive form with perfect information, and $G^{\widetilde{M}} = (N, L_X, (\widetilde{M}_i(G))_{i\in N}, h^G, (t_i(\widetilde{\succcurlyeq}_i))_{i\in N})$ its behavioural-mixed extension. Then, a **perfect behavioural equilibrium** of G is a strategy-profile $\widetilde{m} = (\widetilde{m}_i)_{i\in N} \in \prod_{i\in N} \widetilde{M}_i(G)$ (with $\widetilde{M}_i(G) := \prod_{I_{ij}\in\Pi_i} \widetilde{M}_{ij}$, hence $\widetilde{m}_i = ((\widetilde{m}_{ij}) : I_{ij} \in \Pi_i))$ such that for every $I_{ij} \in \bigcup_{i\in N} \Pi_i$

$\widetilde{m}| \uparrow I_{ij} = ((\widetilde{m}_i| \uparrow I_{ij}))_{i\in N}$ is a Nash equilibrium of the behavioural-mixed extension $G^{\widetilde{M}}_{I_{ij}}$ of the proper subgame $G_{I_{ij}}$ of G (where $(\widetilde{m}_i| \uparrow I_{ij})$ denotes the *continuation behavioural strategy of* $\widetilde{m}_i$ in $G^{\widetilde{M}}$ i.e. the restriction of $\widetilde{m}_i$ to $G^{\widetilde{M}}_{I_{ij}}$).

EXAMPLE It is easily checked that Selten's Horse has both perfect and non-perfect behavioural equilibria.

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